Engineering Analysis with Boundary Elements 29 (2005) 190–194
ENGINEERING ANALYSIS with BOUNDARY ELEMENTS
www.elsevier.com/locate/enganabound

Short communication

Approximating three-dimensional steady-state potential flow problems using two-dimensional complex polynomials

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Received 3 March 2004; revised 19 June 2004; accepted 28 July 2004
Available online 11 January 2005

Abstract

A new advance in the Complex Variable Boundary Element Method, or CVBEM, is its extension to three-dimension (3D). This advance breaks down the barrier of limiting CVBEM models to two-dimensional (2D) problems, and also opens the door to solving 3D potential problems with other 2D numerical analogs. In this paper, a 3D analog is developed using 2D basis functions of the complex analytic polynomial type. Thus, 2D complex polynomials are being applied to 3D potential problems. This new advance may be of interest to those involved in applied mathematics, complex variables, boundary elements, and numerical solution of partial differential equations of the Laplace of Poisson type.

Keywords: CVBEM; Three-dimensional; 3DCVBEM; Potential problems; Boundary elements

1. Introduction

The Complex Variable Boundary Element Method, or CVBEM, has been the subject of several papers and books since the early 1980's. Most recent books are [1,2], which examine two-dimensional theory and applications and the more recent three-dimensional development, or 3DCVBEM. In the latter referenced book, the traditional two-dimensional limitation to use of complex variables is eliminated. Thus, there is now a viable and usable alternative numerical analog available for solving three-dimensional potential flow problems such as diffusion processes, groundwater flow, heat transport, criterion variable spatial distribution (e.g. sediment and air transport and rainfall among many other applications).

In this paper, a new variant of the 3DCVBEM is examined by using sets of two-dimensional complex polynomials instead of the usual CVBEM basis functions of the $(z\text{–}z_j)\ \ln\,(z\text{–}z_j)$ type (see [1]). The basis functions are then applied to numerically solving a steady-state three-dimensional potential flow problem.

2. Mathematical development

Let Ω be a three-dimensional (3D) domain that is oriented in the first octant of the 3D coordinate system (that is, the x, y and z coordinates are all positive). The boundary of Ω is Γ. The steady-state potential values of the state variable, T, are given on Γ by the $T_b(x,y,z)$ function which forms the boundary condition of the 3D Laplace equation, in Ω, of

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0,(x,y,z)\in\Omega\cup\Gamma \quad (1)$$

where $T(x,y,z)$ is the potential function which approaches in value to $T_b(x,y,z)$ as (x,y,z) approaches Γ from inside Ω.

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0955-7997/$ - see front matter © 2004 Published by Elsevier Ltd.
doi:10.1016/j.enganabound.2004.07.004

3. Basis functions used in numerical analog

Consider the two-dimensional (2D) analytic complex polynomials of the form

$$\omega_n(x,y) = (x + \mathrm{i}y)^n \tag{2}$$

where (x,y) are 2D coordinates; $\mathrm{i} = \sqrt{-1}$ and n is a positive integer. For example, for $n=1$, $\omega_1(x,y)=x+\mathrm{i}y$ and for $n=2$, $\omega_2(x,y)=(x^2-y^2)+\mathrm{i}2xy$. The $\omega_n(x,y)$ functions result in a real and imaginary 2D real polynomial harmonic components (e.g. (x^2-y^2) is the real component and $2xy$ is the imaginary component of $\omega_2(x,y)$). Both the real and imaginary components of $\omega_2(x,y)$ satisfy the 2D Laplace equation. Therefore

$$\omega_n(x,y) = \phi_n(x,y) + \mathrm{i}\Psi_n(x,y) \tag{3}$$

where both $\phi_n(x,y)$ and $\psi_n(x,y)$ are 2D potential functions. Consequently, both $\phi_n(x,y)$ and $\psi_n(x,y)$ satisfy both the 2D and the 3D Laplace equation.

In the above equations, (x,y) are 2D coordinates with respect to the orientation of the particular (x,y) plane. Other planes are possible. For example, the familiar (x,z) and (y,z) planes. These other planes can also be used to generate other basis functions that are 2D complex polynomials in their respective planes, but are therefore also 3D potential functions. For example

$$\omega_n(x,z) = (x + \mathrm{i}z)^n \tag{4}$$

results in $\phi_n(x,z)+\mathrm{i}\psi_n(x,z)$, giving two more 2D potential functions.

In order to simplify notation, let $(x,y)^m$ be an arbitrary orthogonal coordinate system in plane m. Another interpretation is to hold the (x,y) plane fixed but then to simply rotate Ω into another arbitrary orientation, and then examine particular properties in the orientation with number m.

Using the plane interpretation, the usual (x,y), (x,z), and (y,z) planes correspond to coordinates $(x,y)^1$, $(x,y)^2$, $(x,y)^3$, respectively.

For a set of M planes (or Ω orientations), there will be M sets of basis functions

$$\omega_{nm}(x,y)^m; \quad m = 1,2,\ldots,M \tag{5}$$

where each $\omega_{nm}(x,y)^m$ is of the form

$$\omega_{nm}(x,y)^m = \phi_{nm}(x,y)^m + \mathrm{i}\Psi_{nm}(x,y)^m \tag{6}$$

For example, for $(x,y)^2$, corresponding to (x,z)

$$\omega_{22}(x,y)^2 = \omega_{22}(x,z) = (x^2 - z^2) + \mathrm{i}2xz \tag{7}$$

Note that there need not be only the three orthogonal planes usually associated with 3D space. Just as there are numerous ways to spin Ω into new orientations, so there are numerous $(x,y)^m$ local coordinate systems.

3.1. Basis functions and the $(x,y)^m$ plane, $\mathbb{P}^m$

An arbitrary $(x,y)^m$ system corresponds to a 2D plane, $\mathbb{P}^m$, that lies in the exterior of Ω. For basis function $\omega_{nm}(x,y)^m$ defined on $\mathbb{P}^m$, the objective is to minimize the difference between the values of $\alpha_{nm}\omega_{nm}(x,y)^m$ and the boundary conditions $T_b(x,y)$ on Γ where α_{nm} is a constant to be determined. Note that $(x,y)^m$ are the coordinates in $\mathbb{P}^m$ whereas (x,y) are the original 2D coordinates.

Therefore, for M orientations of Ω (or M independent planes), the approximation function is

$$A(x,y,z) = \sum_{m=1}^{M}\sum_{n=1}^{N} \alpha_{nm}\omega_{nm}(x,y)^m \tag{8}$$

The MN constants α_{nm} are determined by a generalized Fourier series or least squares minimization of $|T_b(x,y,z) - A(x,y,z)|$ for $(x,y,z)\in\Gamma$. It is noted that the above formulation uses only 2D basis functions, and that these 2D basis functions are complex analytic polynomials (although others are readily possible).

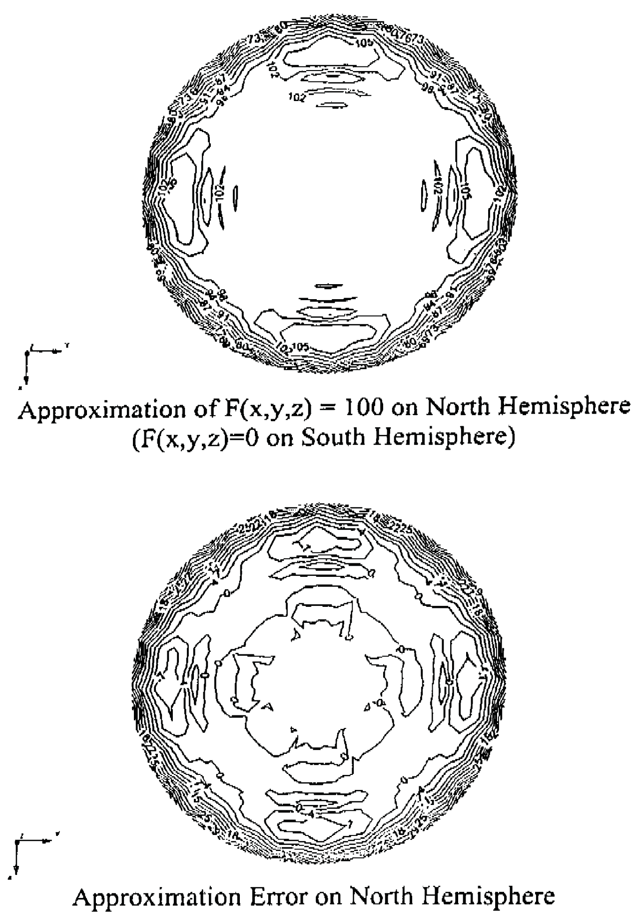


Fig. 1. Application A - approximation results on northern hemisphere.

4. Numerical model

To numerically solve the above minimization problem, the following steps are used (see [2, Chapter 1]):

1. Define ‘integration points’ on Γ by ‘dusting’ the surface of Ω with NINT points (where NINT is the number of integration points). Develop a G vector composed of the (x,y,z) coordinates of each integration point. Thus, G is a NINT$\times$1 column vector. Note that the integration points are only specified on the problem boundary, and not in the interior of the problem domain.
2. Define the NINT$\times$1 boundary condition vector TB by evaluating $T_b(x,y,z)$ at each integration point in the order assembled in the G vector.
3. Decide on M orientations of Ω or, equivalently, M planes for coordinate systems and planes.
4. For each plane in step 3, apply N complex polynomials and evaluate (as accomplished in building the TB vector) at each integration point used in G. The resulting NINT$\times$1 column vectors are denoted as W_{nm}.
5. Using least-squares minimization or a generalized Fourier series approach, minimize the least squares residual between the vector TB and the vector A_{MN} where

$$A_{MN} = \sum_{m=1}^{M}\sum_{n=1}^{N} \alpha_{nm}\omega_{nm} \tag{9}$$

6. The approximation function is then

$$A(x,y,z) = \sum_{m=1}^{M}\sum_{n=1}^{N} \alpha_{nm}\omega_{nm}(x,y)^m \tag{10}$$

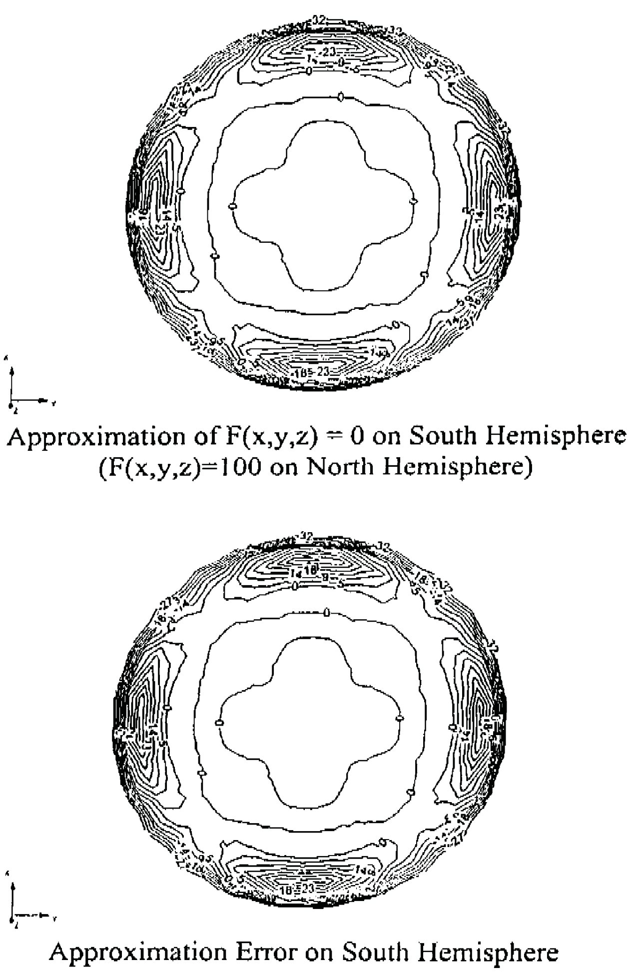


Fig. 2. Application A - approximation results on southern hemisphere.

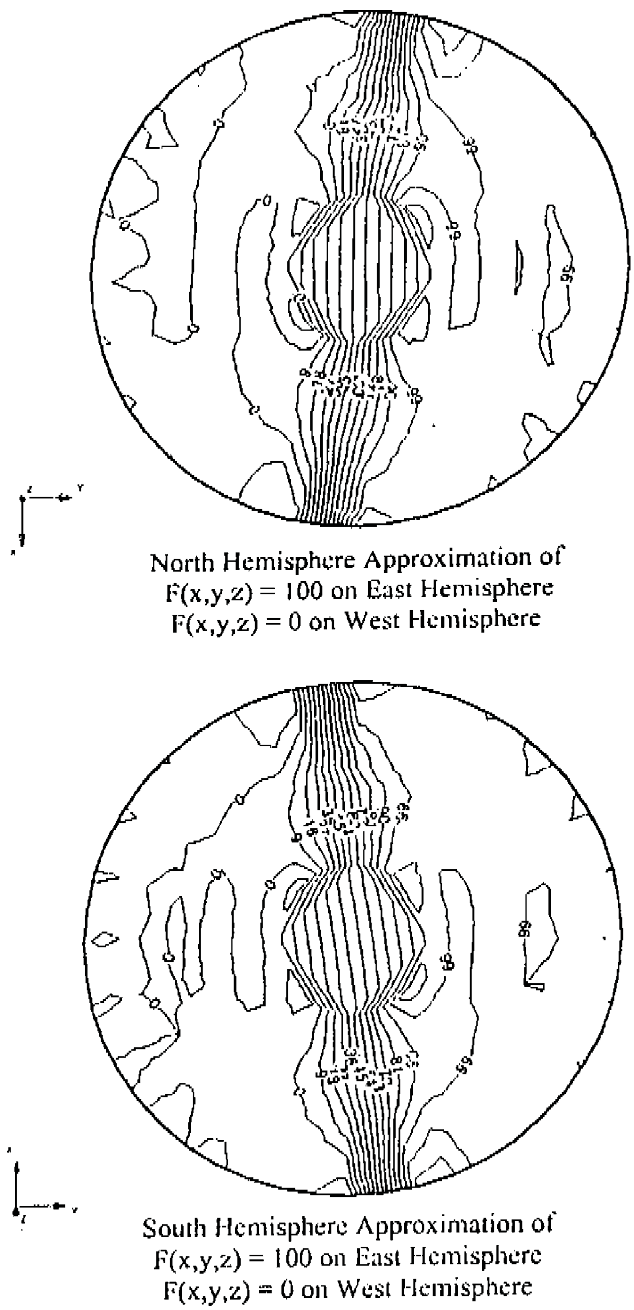
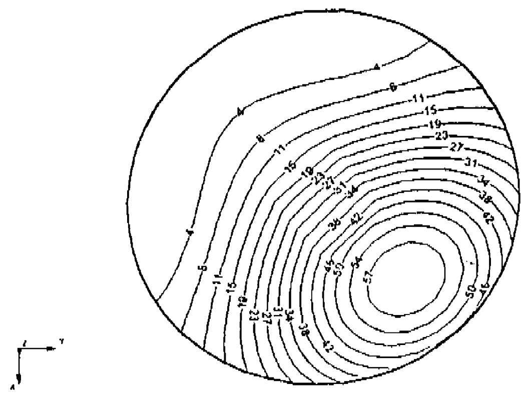
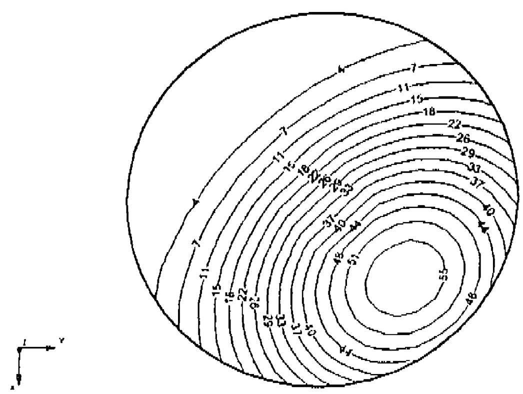


Fig. 3. Application A - approximation results on sphere rotated 90 degrees.

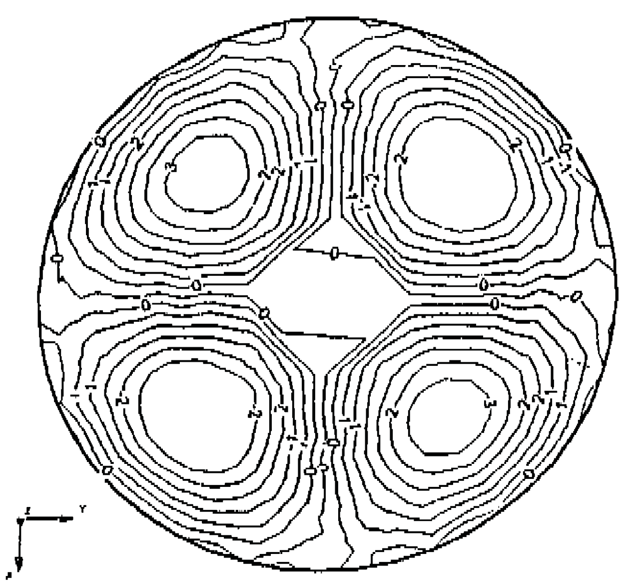


Exact Solution of F(x,y,z)=xyz on North Hemisphere



Approximation of F(x,y,z)=xyz on North Hemisphere

Fig. 4. Application B - exact and approximation isothermals on northern hemisphere.

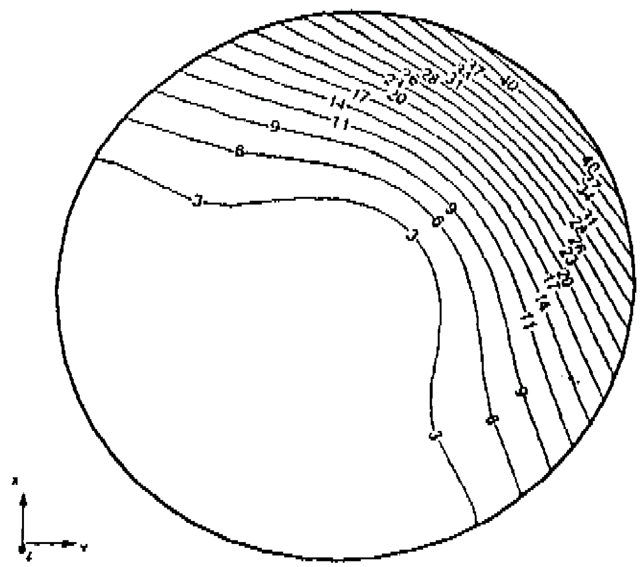


Approximation Error of F(x,y,z)=xyz on North Hemisphere

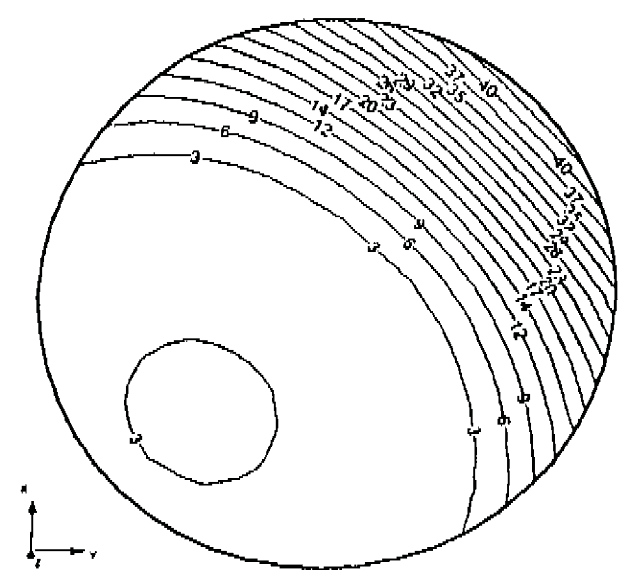
Fig. 5. Application B - error isocontours on northern hemisphere.

5. Example problems

To demonstrate the above methodology, 3D steady-state potential problems were examined using up to five independent planes, and complex polynomials of up to order 5, resulting in a total of 50 2D basis functions (prior to orthonormalization). The five planes used correspond to the usual set of three orthogonal planes in (*x*,*y*), (*y*,*z*), (*x*,*z*) and also two planes oriented at a 45-degree angle with (*x*,*y*) and also the (*x*,*z*) planes, respectively. All planes lie exterior of Ω.



Exact Solution of F(x,y,z)=xyz on South Hemisphere



Approximation of F(x,y,z)=xyz on South Hemisphere

Fig. 6. Application B - exact and approximation isothermals on southern hemisphere.

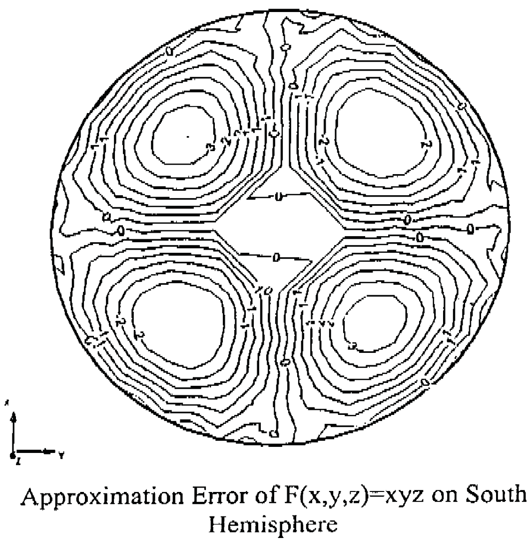


Fig. 7. Application B - error isocontours on southern hemisphere.

As a demonstration, two non-linear potential problems (Applications ‘A’ and ‘B’) are examined that have known analytic solutions and therefore, the approximation error can be readily evaluated on the boundary, Γ, and also in the interior of Ω.

The approximation effort begins by defining a nearly uniform (note: the numerical technique does not require the distribution of integration points to be uniform) distribution of 240 integration points on the boundary, Γ. Again, no such points are defined in the interior of Ω. Therefore, the numerical analog is similar to the usual boundary element method type of applications.

Graphical depictions of the several 3DCVBEM test problems can be seen in Figs. 1–7. Included in the figures are the analytic solutions to the demonstration problems. In addition, isocontour plots of complex polynomial results as well as approximation error are presented.

6. Summary

From the test problems, the use of 2D polynomial harmonic functions (such as obtained from complex variable analytic polynomials) to numerically approximate 3D problems of the LaPlace equations are a viable computational procedure. However, further research is needed to conclude whether such polynomials provide more robust approach for basis functions in the 3DCVBEM numerical technique than by use of the standard 2D CVBEM basis functions.

Acknowledgements

Acknowledgements are paid to the students of the Applied Master program at California State University, who participated in running the 3DCVBEM computer program on numerous test problems and applications in this paper and also the companion paper entitled ‘A Comparison of the 3D Real and Complex Variable Boundary Element Methods (BEM).’

References

[1] Hromadka II TV, Whitley RJ. Advances in the complex variable boundary element method. New York: Springer; 1998. 400 p.
[2] Hromadka II TV. A multi-dimensional complex variable element method. Southampton, England: WIT Press; 2002. 250 p.