Approximating Harmonic Functions on R^n with One Function of a Single Complex Variable

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Let Ω be a bounded open set in R^n, $n > 2$, with $R^n - \bar{\Omega}$ a connected set that is not thin at each point of $\partial\Omega$. Then any solution to a Dirichlet problem for given continuous boundary data on $\partial\Omega$ can be approximated in a simple way by a sum that involves one function $f(z)$ of a single complex variable z; any analytic function $f(z)$ not a polynomial can be used. One consequence of this approximation property is that any harmonic polynomial can be written (exactly, not approximately) as a finite sum involving polynomials in one complex variable. These results reveal an unexpected simplicity in the structure of harmonic functions on R^n. It is a common and simple observation that harmonic analysis is more difficult in three or more dimensions than in two dimensions because you do not have the direct use of the theory of analytic functions of a single complex variable; the results here show that this obvious observation is not correct.

For applications, these approximating sums provide a large collection of functions that can be fit to given boundary data and used for the numerical solution of Dirichlet problems in R^n.

The harmonic functions considered will be real-valued functions of a variable in R^n, typically $x = (x_1, x_2, \ldots, x_n)$, with $n > 2$.

The Walsh-Lebesgue Theorem states that if K is a compact subset of R^2, with $R^2 - K$ connected, then every continuous real-valued function on ∂K can be uniformly approximated by functions of the form Re $P(z)$, $P(z)$ a polynomial in the complex variable z. Theorem 1 generalizes this result to R^n, with a proof that is a modification of the proof given in [1, Corollary 6.3.4, p 173] for R^2.

Theorem 1. *Let Ω be a bounded domain in R^n, $n > 2$, with $R^n - \bar{\Omega}$ connected. Suppose that*

$$R^n - \bar{\Omega} \text{ is not thin at any point of } \partial\Omega. \tag{1}$$

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Given a continuous real-valued function g on $\partial\Omega$ *and* $\epsilon > 0$, *there is a harmonic polynomial* $p(x)$ *satisfying*

$$|g(x) - p(x)| < \epsilon \qquad \text{for all } x \text{ in } \partial\Omega. \tag{2}$$

Conversely, if each continuous real-valued function on $\partial\Omega$ *can be approximated by harmonic polynomials as in* (2), *then* (1) *holds.*

Proof. Let M be the closure of the subspace of $C(\partial\Omega)$ spanned by the harmonic polynomials. To show that $M = C(\partial\Omega)$, it will be shown that if x^* is a continuous linear functional on $C(\partial\Omega)$, which is zero on M, then x^* is zero. The functional x^* can be represented by integration with respect to a regular Borel measure μ on $\partial\Omega$, which can be written as the difference of two non-negative regular Borel measures [2]:

$$\mu = \mu_1 - \mu_2. \tag{3}$$

Consider the Newtonian potentials for the two measures μ_1 and μ_2:

$$P_i(x) = \int_{\partial\Omega} |x - t|^{2-n} d\mu_i(t). \tag{4}$$

Each $P_i(x)$ is harmonic in $R^n - \bar{\Omega}$ and in Ω and is superharmonic in all of R^n.

If $\bar{\Omega}$ is contained in the ball $B(0, r) = \{x : |x| < r\}$ and z is any fixed point with $|z| > 2r$, then for $|x - z| < r$, the function $\phi(t) = |x - t|^{2-n}$ is harmonic in $B(0, r)$. Thus $\phi(t)$ can be written as a sum of harmonic polynomials:

$$\phi(t) = \sum p_j(t), \tag{5}$$

a sum uniformly and absolutely convergent on $\partial\Omega$ [3, corollary 5.34, p 100]. Because $x^*(M) = 0$,

$$\int_{\partial\Omega} p_j(t) d\mu_1(t) = \int_{\partial\Omega} p_j(t) d\mu_2(t), \tag{6}$$

from which it follows that

$$P_1(x) = P_2(x) \qquad \text{for } x \text{ in } R^n - \bar{\Omega}, \tag{7}$$

since the harmonic functions $P_1(x)$ and $P_2(x)$ are equal on the ball $B(z, r)$ contained in the connected set $R^n - \bar{\Omega}$ [3, Theorem 1.27, p 21].

Let ζ be a point in $\partial\Omega$. Because condition (1) holds

$$\liminf\{P_i(x) : x \in R^n - \bar{\Omega}, x \to \zeta\} = P_i(\zeta). \tag{8}$$

[1, p 79] and [4, Theorem 7.2.3, p 199].

Together with (7) this shows that

$$P_1(\zeta) = P_2(\zeta) \qquad \text{for } \zeta \text{ in } \partial\Omega. \tag{9}$$

Let L be the subspace of $C(\partial\Omega)$ consisting of those functions f in $C(\partial\Omega)$ for which there exists a function h harmonic in Ω and continuous on $\bar{\Omega}$ agreeing with f on $\partial\Omega$. The maximum principle shows that the map $f \to h(x)$ is a bounded linear functional on L. Extend this map to a continuous linear functional on all of $C(\partial\Omega)$, which is represented by a regular Borel measure ν_x (harmonic measure, representing the functional on the Shilov boundary [4, 6.4]) for which

$$h(x) = \int_{\partial\Omega} h(t)\,d\nu_x(t). \tag{10}$$

For ζ a point in $R^n - \bar{\Omega}$ (and x in Ω),

$$|x - \zeta|^{2-n} = \int_{\partial\Omega} |t - \zeta|^{2-n}\,d\nu_x(t). \tag{11}$$

Both sides of Equation (11) are non-negative superharmonic functions of ζ in R^n. Since condition (1) is satisfied at each point of the boundary of Ω, Equation (11) holds for ζ in $\partial\Omega$ by the exactly the same argument as was used to pass from (7) to (9).

Then

$$P_1(x) = \int_{\partial\Omega} |x - \zeta|^{2-n}\,d\mu_1(\zeta) \tag{12}$$

$$= \int_{\partial\Omega}\int_{\partial\Omega} |t - \zeta|^{2-n}\,d\nu_x(t)\,d\mu_1(\zeta), \tag{13}$$

and an application of Fubini's theorem shows that

$$P_1(x) = P_2(x) \qquad \text{for all } x \text{ in } \Omega. \tag{14}$$

Hence $P_1(x) = P_2(x)$ holds for all x in R^n, and it follows that

$$\mu_1 = \mu_2 \tag{15}$$

[5, Corollary 1, p 19] or [6, Theorem 6.15, p 112].

Conversely, suppose that Ω is a bounded open set in R^n, with $R^n - \bar{\Omega}$ connected, having the approximation property (2). The approximation property implies the solvability of the Dirichet problem on Ω for continuous boundary data. Given g real-valued and continuous on $\partial\Omega$, let $p_n(x)$ be a sequence of harmonic polynomials satisfying $|g(x) - p_n(x)| < 1/n$ for x in $\partial\Omega$. Then the sequence $\{p_n(x)\}$ converges uniformly on $\bar{\Omega}$ to a function continuous on $\bar{\Omega}$, harmonic on Ω, and equal to g on $\partial\Omega$. Hence every boundary point of Ω is regular and therefore $R^n - \Omega$ is not thin at any point [4, Theorem 7.5.1, p 208]. Further, if h is continuous on $\bar{\Omega}$ and harmonic on Ω, the harmonic polynomial in (2) provides an approximation to h on Ω, by the maximum theorem, which is harmonic in R^n. By [4, Theorem 7.9.5, p 228] or [7, Theorem 1.3, p 11], $R^n - \Omega$ and $R^n - \bar{\Omega}$ are thin at the same points, namely at none. ∎

Recall that a domain Ω in R^n satisfies the Poincaré exterior cone condition at a point ζ in its boundary $\partial\Omega$ if there is an open truncated cone C with vertex ζ and $C - \{\zeta\}$ lying in $R^n - \Omega$ [4, p 186] and [3, p 232]. If Ω is a bounded open set that satisfies an exterior cone condition at every point on its boundary, then the Dirichlet problem with continuous boundary data has a solution, i.e., given a continuous real-valued function g defined on $\partial\Omega$, there is a function h harmonic in Ω and continuous on $\bar{\Omega}$ with $h(x) = g(x)$ for all x in $\partial\Omega$ [4, Chapter 6] and [3, Chapter 11].

Corollary 1. *Let Ω be a bounded domain in R^n, with $R^n - \bar{\Omega}$ connected. Suppose that Ω satisfies an exterior cone condition at each point of $\partial\Omega$. Given a continuous real-valued function g on $\partial\Omega$ and $\epsilon > 0$, there is a harmonic polynomial $p(x)$ satisfying*

$$|g(x) - p(x)| < \epsilon \qquad \text{for all } x \text{ in } \partial\Omega. \tag{16}$$

Proof. Given ζ in $\partial\Omega$, let C be an open cone with vertex at ζ and $B(\zeta, r)$ be a ball for which $B(\zeta, r) \cap C$ lies in $R^n - \Omega$. Then

$$B(\zeta, r) \cap C \subset R^n - \bar{\Omega}. \tag{17}$$

Since the truncated cone is not thin at ζ [4, Theorem 6.6.15, p 185] or [3, Lemma 11.16, p 232], condition (1) of Theorem 1 is satisfied. ∎

Lemma 1. (ii)

(i) *Let h be a harmonic function of two variables defined on an open set U in R^2, and let a and b be two perpendicular vectors, $a \cdot b = 0$, of equal length $|a| = |b|$ in R^n. Then $h(a \cdot x, b \cdot x)$ is a harmonic function for x in R^n and $(a \cdot x, b \cdot x)$ in U.*

(ii) *Let f be analytic on a disk $D(z_1, r) = \{z : |z - z_1| < r\}$. If f is not a polynomial on D, there is a point z_0 in D where every derivative of f is nonzero*:

$$f^{(j)}(z_0) \neq 0 \qquad \text{for } j = 0, 1, 2, \ldots. \tag{18}$$

Proof. Set $H(x) = h(a \cdot x, b \cdot x)$, and (i) follows from the computation of $\nabla^2 H(x)$:

$$h_{11}(a \cdot x, b \cdot x) \sum_1^n a_j^2 + h_{22}(a \cdot x, b \cdot x) \sum_1^n b_j^2 + 2h_{12}(a \cdot x, b \cdot x) \sum_1^n a_j b_j. \tag{19}$$

Let $D_j = \{z \text{ in } D: f^{(j)}(z) = 0\}$, $j = 0, 1, 2, \ldots$. If (ii) is not true, then $D = \cup D_j$ and any closed uncountable subset F of D intersects at least one set D_m in an infinite set with a limit point in D; by the identity theorem $f^{(m)}$ is identically zero in D and f is a polynomial. (See [8] or [2, ex. 2, p 227] or [9, p 53]). ∎

Theorem 2. *Let $f(z)$ be a function analytic in a disk $D(z_0, \rho)$, where it is not a polynomial; using Lemma 1 we can suppose that (18) holds. Let Ω be a domain in R^n, satisfying the hypotheses of Theorem 1; and let $g(x)$ a real-valued function on $\partial\Omega$ and $\epsilon > 0$ be given. There are complex constants α_j and vectors a^j and b^j in R^n, $j = 1, 2, \ldots, N$ with*

$$a^j \cdot b^j = 0 \qquad \text{and} \qquad |a^j| = |b^j| < r, \tag{20}$$

r chosen so that

$$|a^j \cdot x + ib^j \cdot x| < \rho, \qquad \text{for all } x \text{ in } \bar{\Omega}, \tag{21}$$

so that the corresponding function

$$h(x) = \mathrm{Re} \sum_{j=1}^{N} \alpha_j f(z_0 + (a^j \cdot x + ib^j \cdot x)), \tag{22}$$

defined and harmonic on $\bar{\Omega}$, satisfies

$$|h(x) - g(x)| < \epsilon \qquad \text{for } x \text{ in } \partial\Omega. \tag{23}$$

Consequently for all x in $\bar{\Omega}$, $h(x)$ is within ϵ of the exact solution $u(x)$ to the Dirichlet problem with boundary data g.

Proof. By Theorem 1 it will suffice to consider boundary data given by a harmonic polynomial $p(x)$. Since a harmonic polynomial $p(x)$ of degree k can be written as a finite sum of harmonic polynomials homogeneous of degree j, $j = 0, 1, \ldots, k$ [3, theorem 1.31, p 25], it will be enough to approximate each of these homogeneous polynomials and so we can also assume $p(x)$ to be homogeneous of degree m.

The first step in the proof will be to find a specific approximation of the general type displayed in (22). This will be done by deviating from the customary approach of analyzing harmonic functions in terms of their values on spheres, and instead considering the cube:

$$K = \{x : 0 \le x_1 \le \pi, 0 \le x_2 \le \pi, \ldots, 0 \le x_n \le \pi\}, \tag{24}$$

with center

$$c_0 = \left(\frac{\pi}{2}, \frac{\pi}{2}, \ldots, \frac{\pi}{2}\right). \tag{25}$$

Let

$$q(x) = p(x + c_0) \tag{26}$$

and consider the Dirichlet problem on K with continuous boundary values given by restricting $q(x)$ to ∂K.

The classical method of separation of variables applies to such a problem. A particular solution u_F will be found that is equal to $q(x)$ on one face F of K and is zero on all the other faces; the solution to the general problem is then the sum of such functions, one for each face.

Consider the face F given by

$$F = \{x = (\pi, x_2, x_3, \ldots, x_n) : 0 \le x_2 \le \pi, 0 \le x_3 \le \pi, \ldots, 0 \le x_n \le \pi\}. \tag{27}$$

By separation of variables the particular solution $u_F(x)$ is obtained as a sum over the multi-index $\alpha = (\alpha_2, \ldots, \alpha_n)$:

$$u_F(x) = \sum_{\alpha} c_\alpha \sinh(k_\alpha x_1)\sin(\alpha_2 x_2) \cdots \sin(\alpha_n x_n), \tag{28}$$

with

$$k_\alpha^2 = \alpha_2^2 + \alpha_3^2 + \cdots + \alpha_n^2, \tag{29}$$

where the coefficients are computed so that when restricted to F, this sum is the multivariable Fourier series of $q(x)$. Because $q(x)$ is a polynomial, the convergence of this series can be analyzed using the theory of Fourier series in one variable. A typical term in $q(x)$ has the form $x_2^{m_2} x_3^{m_3} \cdots x_n^{m_n}$, with Fourier coefficient

$$(2/\pi)^{n-1} \int_0^\pi \cdots \int_0^\pi x_2^{m_2} \cdots x_n^{m_n} \sin(\alpha_2 x_2) \cdots \sin(\alpha_n x_n) dx_2 \cdots dx_n, \tag{30}$$

which equals the product

$$\frac{2}{\pi} \int_0^\pi x_2^{m_2} \sin(\alpha_2 x_2) dx_2 \cdots \frac{2}{\pi} \int_0^\pi x_n^{m_n} \sin(\alpha_n x_n) dx_n. \tag{31}$$

Hence on F, a partial sum of the Fourier series for $x_2^{m_2} x_3^{m_3} \cdots x_n^{m_n}$ is

$$\sum_{\alpha_2=1}^{N_2} \sum_{\alpha_3=1}^{N_3} \cdots \sum_{\alpha_n=1}^{N_n} c'_\alpha \sin \alpha_2 x_2 \cdots \sin \alpha_n x_n, \tag{32}$$

which equals

$$\sum_{\alpha_2=1}^{N_2} c''_{\alpha_2} \sin(\alpha_2 x_2) \sum_{\alpha_3=1}^{N_3} c''_{\alpha_3} \sin(\alpha_3 x_3) \cdots \sum_{\alpha_n=1}^{N_n} c''_{\alpha_n} \sin(\alpha_n x_n), \tag{33}$$

the product of the partial sums of the Fourier series for $x_2^{m_2}, \ldots, x_n^{m_n}$. Hence for $x_1 = \pi$, the series (28) converges boundedly to $q(x)$ for $0 < x_2 < \pi, \ldots, 0 < x_n < \pi$ [10, Theorem 45, p 32], and to zero if any of $x_2, \ldots, x_n$ are either 0 or π. Summing over all the faces of K defines the putative solution

$$u(x) = \sum \{u_F(x) : F \text{ a face of } K\}. \tag{34}$$

If this solution u were equal to $q(x)$ on all of ∂K, it would be immediate that $u(x)$ and $q(x)$ are equal in the interior of K. However, $u(x) = q(x)$ holds for certain only on $\partial K - E$, E denoting the edges of K. What is needed to conclude that $u(x)$ and $q(x)$ are equal in the interior of K is a Phragmén-Lindelöf theorem for the Laplacian [11, Corollary p 99 and pp 99–102]. The hypotheses of this theorem require that u and q be bounded and that a function $w(x)$ exists satisfying $w(x) > 0$ on K, $w(x)$ harmonic in the interior of K, and

$$\lim\{w(x) : x \to y, x \text{ in the interior of } K\} = +\infty \text{ for } y \text{ in } E. \tag{35}$$

For the specific edge

$$E_1 = \{x = (0, 0, x_3, x_4, \ldots, x_n)\}, \tag{36}$$

consider

$$w_1(x) = \int_0^{\pi} \cdots \int_0^{\pi} |x - (0, 0, t_3, t_4, \ldots, t_n)|^{2-n} dt_3 dt_4 \cdots dt_n. \tag{37}$$

The function $w_1(x)$ is harmonic on $R^n - E_1$, and by evaluating the limiting integral in polar coordinates is seen to tend to $+\infty$ as x approaches any point on E_1. Adding up such functions, one for each edge of K, gives the desired function $w(x)$ and the above mentioned theorem applies showing that $u(x) = q(x)$ in the interior of K.

Examination of the coefficients in the series for $u(x)$ shows that it converges uniformly on compact subsets of the interior of K, so in particular on the closed ball of radius one with center at the center of K,

$$\bar{B} = \overline{B(c_o, 1)}. \tag{38}$$

Thus, given $\epsilon > 0$, there is a partial sum v of the series for u with

$$|v(x) - q(x)| < \epsilon \qquad \text{for } x \text{ in } \bar{B}. \tag{39}$$

The terms of v can be written in a different form using trigonometric identities that are expressed as sums over all possible values of $\epsilon_2, \epsilon_3, \ldots, \epsilon_m$, where each ϵ_j is either $+1$ or -1:

$$\sin(\theta_1)\sin(\theta_2) \cdots \sin(\theta_m) = \frac{-1}{2^{m-1}} \sum \epsilon_2 \cdots \epsilon_m \sin(\theta_1 + \epsilon_2\theta_2 + \cdots + \epsilon_m\theta_m) \tag{40}$$

for m odd, and

$$\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_m) = \frac{-1}{2^{m-1}}\sum \epsilon_2\cdots\epsilon_m\cos(\theta_1 + \epsilon_2\theta_2 + \cdots + \epsilon_m\theta_m) \tag{41}$$

for m even. Substituting these expressions in the terms of (28) gives a new series with typical term either a constant times

$$\sinh(k_\alpha x_1)\sin(\alpha_2 x_2 + \epsilon_3\alpha_3 x_3 + \cdots + \epsilon_n\alpha_n x_n) \tag{42}$$

or a constant times

$$\sinh(k_\alpha x_1)\cos(\alpha_2 x_2 + \epsilon_3\alpha_3 x_3 + \cdots + \epsilon_n\alpha_n x_n). \tag{43}$$

The R^n vectors

$$a = (k_\alpha, 0, 0, \ldots, 0) \tag{44}$$

and

$$b = (0, \alpha_2, \epsilon_3\alpha_3, \ldots, \epsilon_n\alpha_n) \tag{45}$$

are obviously perpendicular and have the same length by (29). Therefore each term in the sum u_F can be written as a sum of functions of the form

$$h(a\cdot x, b\cdot x), \tag{46}$$

where h is a function of two variables harmonic on R^2 and a and b are perpendicular vectors in R^n of the same length. This is also true when separation of variables is applied to the lower face $\{x : x = (0, x_2, x_3, \ldots, x_n)\}$, the only difference in form is that $\sinh(\alpha_k x_1)$ is replaced by $\sinh(\alpha_k(x_1 - \pi))$.

Thus, the partial sum v, which satisfies (39), can be written as a finite sum

$$v(x) = \sum_1^N h_j(a^j\cdot x, b^j\cdot x), \tag{47}$$

each h_j a function of two variables harmonic on all of R^2, and each pair a^j and b^j perpendicular vectors of the same length in R^n.

From (26)

$$|p(x) - v(c_0 + x)| < \epsilon \qquad \text{for } |x| \leq 1. \tag{48}$$

Translation by c_0 does not change the form of the function v as each term in the series representing $v(x + c_0)$ is

$$H_j(a^j\cdot x, b^j\cdot x) = h_j(a^j\cdot x + a^j\cdot c_0, b^j\cdot x + b^j\cdot c_0), \tag{49}$$

where H_j is a function of two variables, harmonic on R^2, and

$$\left| p(x) - \sum_1^N H_j(a^j \cdot x, b^j \cdot x) \right| < \epsilon \qquad \text{for } |x| \leq 1. \tag{50}$$

Since Ω is bounded, $\bar{\Omega}$ is contained in a ball $B(0, r_1)$. Using the fact that the polynomial $p(x)$ is homogeneous of degree m,

$$\left| p(x) - r_1^m v\left(\frac{x}{r_1}\right) \right| \leq r_1^m \epsilon \qquad \text{for } |x| \leq r_1. \tag{51}$$

Therefore $p(x)$ can be approximated to arbitrary accuracy on $\bar{\Omega}$ by functions of the form $r_1^m v(x/r_1)$, with no scaling necessary if $r_1 < 1$. The change of variable in $v(x)$ to x/r_1 can be accomplished by writing a^j/r_1 and b^j/r_1 for a^j and b^j; and when this is done (50) can be assumed to hold for $|x| \leq r_1$ and therefore for x in $\bar{\Omega}$.

Let

$$r_2 = 2 \sup\{|x|\,|a^j| : x \text{ in } \Omega, j = 1, 2, \ldots, N\}, \tag{52}$$

so that for x in Ω and all a^j and b^j appearing in the sum (50), the R^2 vector $(a^j \cdot x, b^j \cdot x)$ has length bounded by r_2. For each term H_j in the sum (50), consider the Dirichlet problem in R^2 with domain the disk $D(0, r_2)$ and boundary values those obtained by restricting H_j to $\partial D(0, r_2)$. By [8, Theorem 2], given $\epsilon > 0$ there is a number $r > 0$ and a finite number N_j of complex constants c_k^j and real numbers λ_k^j, $|\lambda_k^j| < r$, with

$$\left| H_j(\mathrm{Re}(z), \mathrm{Im}(z)) - \mathrm{Re} \sum_{k=1}^{N_j} c_k^j f(z_0 + \lambda_k^j z) \right| < \frac{\epsilon}{N} \tag{53}$$

holding for all z in $\partial D(0, r_2)$, and therefore in $D(0, r_2)$ since both H_j and the approximating sum are harmonic in $D(0, r_2)$. Adding up these approximations for all the H_j terms in the sum for $p(x)$ gives

$$\left| p(x) - \mathrm{Re} \sum_{j=1}^{N} \sum_{k=1}^{N_j} c_k^j f(z_0 + (\lambda_k^j(a^j \cdot x + ib^j \cdot x) \right| < \epsilon \tag{54}$$

for all x in $D(0, r_2)$ and so for all x in $\bar{\Omega}$. Replacing a^j and b^j by $\lambda_k^j a^j$ and $\lambda_k^j b^j$ gives an approximation as in (22). ∎

The proof shows that the vectors a^j and b^j appearing in the approximating sum may be taken to have a special form where each a^j has only one nonzero coordinate.

Theorem 3. *The vector space of polynomials on R^n, which are harmonic and homogeneous of degree k, has a basis of the form*

$$\{\text{Re}(a \cdot x + ib \cdot x)^k, \text{Im}(a \cdot x + ib \cdot x)^k, a, b \text{ in } R^n, a \cdot b = 0, |a| = |b|\} \tag{55}$$

Consequently a harmonic polynomial of degree m can be written as a linear combination of the polynomials in (55) *using* $k = 0, 1, \ldots, m$.

Proof. Let p be a harmonic polynomial of degree less than or equal to m, and let $\epsilon > 0$ be given. Taking the analytic function f to be entire in Theorem 2, there is a function $h(x)$ harmonic on R^n,

$$h(x) = \sum_{j=1}^{N} h_j(a^j \cdot x, b^j \cdot x), \tag{56}$$

where each h_j is a function of two variables, harmonic on R^2, and a^j and b^j are perpendicular vectors in R^n of the same length, satisfying

$$|p(x) - h(x)| < \epsilon \qquad \text{for } x \text{ in } B(0, 2). \tag{57}$$

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with $|\alpha| = m + 1$, there is a constant C_α, depending only on α and $r = 2$, so that the Cauchy estimates [3, p 33]

$$|D^\alpha(p(x) - h(x))| = |D^\alpha h(x)| \leq \epsilon \frac{C_\alpha}{2^{m+1}} \tag{58}$$

hold for all x in $B(0, 2)$. The function $h(x)$ can be expanded in a Taylor's series about 0 giving

$$h(x) = q(x) + R_m(\xi), \qquad \xi \text{ on the line joining } x \text{ to } 0, \tag{59}$$

$q(x)$ the Taylor polynomial of degree less than or equal to m, with remainder of the form

$$R_m(\xi) = \frac{1}{(m+1)!} \sum_{|\alpha|=m+1} \binom{m+1}{\alpha} D^{\alpha_1} \cdots D^{\alpha_n} h(\xi) x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \tag{60}$$

Since the constants in (58) are known, by a choosing the correct multiple of ϵ in inequality (57) it is possible to have

$$|R_m(\xi)| < \epsilon \tag{61}$$

holding for all possible ξ in $B(0, 2)$. Then

$$|p(x) - q(x)| < \epsilon \qquad \text{for } |x| < 2. \tag{62}$$

Each function $h_j(y_1, y_2)$, harmonic in two variables in all of R^2, can be expanded in a Taylor's series

$$h_j(y_1, y_2) = q_j(y_1, y_2) + R_m^{(j)}(\zeta_1^j, \zeta_2^j) \tag{63}$$

for some point (ζ_1^j, ζ_2^j) lying on the line in R^2 joining the point (y_1, y_2) to 0, with

$$q_j(y_1, y_2) = \sum_{|\alpha| \leq m} \frac{D^\alpha h_j(0)}{\alpha!} y^\alpha \tag{64}$$

a harmonic polynomial [3, pp 23–24] of degree less than or equal to m. Since $q_j(y_1, y_2)$ is harmonic, there is polynomial $Q_j(z)$ in the complex variable z, with

$$q_j(y_1, y_2) = \operatorname{Re} Q_j(y_1 + iy_2). \tag{65}$$

The remainder is

$$R_m^{(j)}(\zeta_1^j, \zeta_2^j) = \frac{1}{(m+1)!} \sum_{k=0}^{m+1} \binom{m+1}{k} D_1^k D_2^{m+1-k} h_j(\zeta_1^j, \zeta_2^j) y_1^k y_2^{m+1-k}. \tag{66}$$

Note that for (y_1, y_2) in a bounded set, the Cauchy inequalities show that the mixed partial derivatives of h, which multiply the terms $y_1^k y_2^{m+1-k}$ in (66), are bounded. At the point $(a^j \cdot x, b^j \cdot x)$, this remainder is

$$\frac{1}{(m+1)!} \sum_{k=0}^{m+1} \binom{m+1}{k} D_1^k D_2^{m+1-k} h_j(\zeta_1^j, \zeta_2^j)(a^j \cdot x)^k (b^j \cdot x)^{m+1-k}. \tag{67}$$

The function h can then be written as

$$h(x) = \hat{q}(x) + \sum_{j=1}^{N} R_m^{(j)}(\zeta_1^j, \zeta_2^j) \tag{68}$$

with $\hat{q}(x)$ a harmonic polynomial of degree less than or equal to m, which is a linear combination of the functions given in (55) for $k = 0, 1, \ldots, m$. Since

$$\frac{|a^j \cdot x|^k |b^j \cdot x|^{m+1-k}}{|x|^m} \to 0 \qquad \text{as } x \to 0, \tag{69}$$

Equation (67) shows that $\hat{q}$ must be the mth degree Taylor polynomial of h [12, Theorem 7.4, p 135],

$$\hat{q}(x) = q(x), \tag{70}$$

and

$$|p(x) - \hat{q}(x)| < \epsilon \qquad \text{for } x \text{ in } B(0, 2). \tag{71}$$

Let M be the subspace of all harmonic polynomials of degree less than or equal to m, taken with the norm

$$\|p\| = \sup\{|p(x)| : |x| \leq 1\}. \tag{72}$$

The space M is finite dimensional, being a subspace of the polynomials of degree less than or equal to m, and so has a compact unit sphere. Thus there are a finite number of polynomials in M of norm less than or equal to one, $p_1, p_2, \ldots, p_r$, with

$$\bigcup_{j=1}^{r} B(p_j, 0.5) \supseteq S_M, \qquad S_M \text{ the closed unit ball in } M. \tag{73}$$

For each p_j, choose $\widehat{q_j}$ as in (68), a harmonic polynomial which is a linear combination of the functions displayed in (55), for $k = 0, 1, \ldots, m$, and therefore itself belonging to M, with

$$\|p_j - \widehat{q_j}\| < \tfrac{1}{2}. \tag{74}$$

Let L be the subspace of M spanned by $\widehat{q_1}, \widehat{q_2}, \ldots, \widehat{q_r}$:

$$L = sp(\widehat{q_1}, \widehat{q_2}, \ldots, \widehat{q_r}). \tag{75}$$

It cannot be that L is a proper subspace of M, for if were there would be a p in M with

$$\|p\| = 1 = dist(p, L), \tag{76}$$

contradicting (73) and (74).

It has been shown that a harmonic polynomial of degree m can be written as a sum of homogeneous polynomials of degree k, taken from the set (55) for various vectors a and b and various values of $k = 0, 1, \ldots, m$. If a polynomial $p(x)$ is expanded as a sum of polynomials $p_j(x)$, each homogeneous of degree j,

$$p(x) = \sum_{j=1}^{m} p_j(x), \tag{77}$$

this expansion is unique and each homogeneous polynomial $p_j(x)$ is harmonic if $p(x)$ is harmonic [3, pp 23–24]. Hence if $p(x)$ is harmonic and homogeneous of degree m, it must equal the sum of the terms in (77), which are themselves homogeneous of degree m, and the theorem follows. ■

NOTE ADDED IN PROOF

Stephen Gardiner notes that Theorem 1 is contained in Theorem 1.15 [7] and Theorem 7.9.7 [4].

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