Engineering Analysis with Boundary Elements 30 (2006) 1019

ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

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Preface

Complex variable boundary element method

Since its inception by Hromadka and Guymon in 1983, the Complex Variable Boundary Element Method or CVBEM has been the subject of several theoretical developments as well as numerous exciting applications. The CVBEM is a numerical application of the Cauchy Integral theorem of complex variables, to two-dimensional potential problems involving Laplace or Poisson equations. This attribute of the CVBEM is a distinct advantage over other numerical techniques that develop only an inexact approximation of the Laplace equation.

During the years 2000 and 2001, publications that extended the two-dimensional CVBEM to three or even higher dimensions appeared, including a book published by the WIT Press (A multi-dimensional Complex Boundary Element Method, T.V. Hromadka II, ISBN:1-85312-908-9) that provides a complete presentation of the development and application of the multi-dimensional CVBEM. Theorems that proved the convergence of the multi-dimensional CVBEM to the solution of the problem and also that provide the pathway towards modeling three or more dimensions using two-dimensional basis functions of real or complex variables were developed.

In this Special Issue of Engineering Analysis with Boundary Elements, several advances and new contributions to the CVBEM are presented. Also included is a discussion of possible future research topics. One important future research topic is the extension of complex variable analytic function theory from two-dimensions to multiple dimensions.

In my own review of these papers, I see a bright future for increased research activity in the CVBEM and extensions of those results to the real variable boundary element methods.

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doi:10.1016/j.enganabound.2006.08.001

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ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

ELSEVIER

Engineering Analysis with Boundary Elements 30 (2006) 1020–1024

www.elsevier.com/locate/enganabound

Theoretical developments in the complex variable boundary element method

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Received 23 May 2006; accepted 18 August 2006

Available online 10 October 2006

Abstract

A review is given of complex variable based numerical solutions, CVBEM methods, for Dirichlet potential problems in two and higher dimensions.

Keywords: CVBEM; Numerical solutions to Dirichlet problems; Complex variable approximate methods

1. Introduction

The foundations of the complex variable boundary element method (CVBEM) are of particular interest because of its usefulness in the numerical solution of a variety of applied problems [1,2]. This paper is a review of the theoretical developments of the CVBEM which show that problems of Dirichlet-type are solvable to within an arbitrarily small preassigned error by use of this method, which extends the method to a larger class of planar problems with a more general type of approximating functions, and which extends the method to problems in higher dimensions.

2. The Dirichlet problem

Consider a basic Dirichlet problem: Let Ω be a bounded simply connected domain in the complex plane with boundary Γ and let g be a real-valued continuous function defined on Γ. The unique solution to this Dirichlet problem is a function $u(z)$, harmonic in Ω and continuous on $\Omega \cup \Gamma$ with $u(z) = g(z)$ for all z on Γ.

The CVBEM method furnishes an approximate solution to this problem if given $\varepsilon > 0$, there is a function $h(z)$ of "CVBEM-type" which is analytic in Ω, continuous on $\Omega \cup \Gamma$, and satisfies

$$|\mathrm{Re}\, h(z) - g(z)| < \varepsilon \quad \text{for all } z \text{ on } \Gamma, \tag{1}$$

for then $\mathrm{Re}\, h(z)$ is harmonic in Ω and, by the Maximum Theorem, is within ε of the exact solution on all of Ω.

This was shown to be the case in [1, Chapter 6] under very restrictive conditions on the boundary Γ and on the boundary value function g. In [3] it was shown to hold for the class of CVBEM functions described below and for a piece-wise smooth boundary: a simple closed curve of finite length with at most a finite number of corner points which are not cusps. This result easily extends from the supremum norm implicit in (1) to integral norms on Γ which are useful in treating discontinuous boundary functions.

The original class of CVBEM functions have the form

$$h(z) = a_0 + a_1 z + \sum_{n=1}^{N} c_n (z - \beta_n) \log_{\beta_n}(z - \beta_n), \tag{2}$$

where $a_0, a_1, c_1, \ldots, c_n$ are complex numbers and the $\beta_1, \ldots, \beta_n$ are points on Γ.

The subscript β occurring in the complex logarithms $\log_\beta(z - \beta)$ in sum (2) indicates a basic aspect of these logarithms: for such a logarithm to be analytic in Ω requires a precise specification of its domain, which means defining a branch cut, i.e. a non-self intersecting curve lying

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doi:10.1016/j.enganabound.2006.08.002

in the complement of $\Omega \cup \Gamma$, except for one end at a point β on Γ, and joining that point to infinity. This is discussed in detail in [4]. Although basically an elementary matter, anyone doing numerical calculations using $f(z) = \log_\beta(z-\beta)$ should as a check numerically integrate $f(z)$ around some closed curves lying in Ω and see that the answers are zero, as it is not entirely trivial to get the argument of the logarithm correctly specified.

The proof in [3], which is technically complicated and non-constructive, uses singular integrals of the form

$$\int_\Gamma \frac{u(z)}{z-\beta}\,\mathrm{d}z$$

whose definition is analogous to that of the Cauchy principle value on the real line in that one integrates over all of Γ excepting an arc cut out of the curve by a small circle of radius r centered at β, and then lets r tend to zero. The use of these integrals gives a rigorous way of deriving the CVBEM function (1) from the Cauchy integral formula, generalized to the case where β lies not in the domain but on the curve Γ. This is discussed, for example, in [4–6]. The use of the CVBEM to give an approximate solution requires a numerical determination of the coefficients $a_0, a_1, c_1, \ldots, c_n, \beta_1, \ldots, \beta_n$ in Eq. (2).

The choice of the $\{\beta_j\}$ has been done on an intuitive basis, putting more points where the Γ has sharper curvature or where the boundary function $g(z)$ changes more rapidly. Since the problem is basically one of integration, it is likely that there are ways of choosing points, analogous to Gaussian integration, which give better results, but this has not been investigated.

Basically, two ways of choosing the $a_0, a_1, c_1, \ldots, c_n$ have been used.

One way is to force the function to interpolate at the points $\{\beta_j\}$. This has been used in connection with the idea of an approximate boundary [1,2] which is a way of producing a domain Ω', with boundary Γ', which is geometrically close to Ω and on which "the same" boundary value problem has an exact CVBEM solution. This idea has not been discussed in a rigorous way but should be studied as it gives a concrete way of visualizing the error in the numerical approximation. For example, if the domain Ω' is within the construction tolerances for Ω there is little point in trying for a solution closer to the true solution for Ω. There are variations of this interpolation method which do not require exact interpolation at the β's.

The second way, which is the simplest in practice, is, after the β's have been chosen, find the least-squares fit of Eq. (2) to the boundary data.

Since the error bound most often cited is the uniform bound of (1), it would seem that coefficients chosen to minimize that norm, rather than the L^2 norm corresponding to least squares, would give a better uniform fit; some unpublished work did show an improvement based on sample calculations, but no comparisons have been published.

3. The Dirichlet problem again

In [7] another proof is given, more direct than the proof of [4], showing that the same Dirichlet problem (1) of Section 2 above has an approximate CVBEM solution of the type displayed in (2). While the main theorem is stated for a twice continuously differentiable boundary Γ, that requirement can be weakened at the cost of making certain estimates more complicated. Many steps in this new proof are constructive in the sense that an estimate can be given for how many β's in sum (2) are needed at certain stages of the proof .

One question for all the methods discussed here is: given $\varepsilon > 0$, what is a bound on the number of terms N in (2) required to obtain (1)? Theorem 1 of [7] can be used to answer this question only under the restrictive hypotheses that the harmonic function which solves the Dirichlet problem is the real part of a function which is analytic on a larger open set containing $\Omega \cup \Gamma$. An answer to the general question would provide a way of quantifying the practical efficiency of the CVBEM; if the accuracy required in a specific Dirichlet problem is $\varepsilon = .001$, can you estimate whether that will take a practical number $n = 10$ terms in the sum or an impractical $n = 10^6$ before you begin the calculations?

A device used in the proof is to move the nodes $\{\beta_1, \ldots, \beta_N\}$ slightly outside the domain, which a technique that has been used in computations.

4. Mixed boundary value problems

Mixed boundary value problems are variations of the Dirichlet problem where a linear combination of the potential and its normal derivative are prescribed on the boundary. One source of mixed boundary value problems is a heat flow problem in which the temperature is prescribed on part of the boundary while the remaining portion of the boundary is insulated. Another source is a problem concerning fluid flow around an obstacle; the obstacle is represented by a hole in the domain Ω which, unlike the domains considered up till now, is then not simply connected.

Consider a planar fluid flow problem with

$$H(z) = \phi(z) + \mathrm{i}\,\psi(z) \tag{3}$$

being the complex potential for the flow with $\phi(z)$ the velocity potential. Applying the Cauchy–Riemann equations

$$H'(z) = \phi_x(z) - \mathrm{i}\phi_y(z) \tag{4}$$

and the right-hand side is the conjugate of the velocity field $\nabla\phi$.

For simplicity consider the case where the boundary Γ is divided into two disjoint pieces Γ_1 and Γ_2, and it is required to have the potential $\phi(z)$ equal to a given real-valued function $g_1(z)$ on Γ_1 and the normal derivative equal to a real-valued function $g_2(z)$ on Γ_2:

$$\phi(z) = g_1(z) \quad \text{for } z \text{ on } \Gamma_1, \tag{5}$$

$$\frac{\partial\phi(z)}{\partial n} = g_2(z) \quad \text{for } z \text{ on } \Gamma_2. \tag{6}$$

Note that the normal derivative of ϕ is the dot product $\nabla\phi(z)\cdot n(z)$ where $n(z)$ denotes the outward pointing normal to Γ at z. To be able to approximate both ϕ and its gradient it is necessary to broaden the class of CVBEM functions. Instead of the usual functions

$$f_\beta(z) = (z-\beta)\log_\beta(z-\beta), \tag{7}$$

consider

$$F_\beta(z) = \frac{(z-\beta)^2}{2}\log_\beta(z-\beta) - \frac{(z-\beta)^2}{4}, \tag{8}$$

the derivative of $F_\beta(z)$ being $f_\beta(z)$. The appropriate class of functions for a mixed boundary value problem turns out to be

$$H(z) = a_0 + a_0' z + a_o'' z^2 + \sum_{n=1}^{N} a_k F_{\beta_k}(z). \tag{9}$$

The first theorem in [8] applies to the same class of domains as in [7], Ω a simply connected domain with a piecewise continuously differentiable boundary of finite length with at most a finite number of corners which are not cusps. It states that if ϕ is a function harmonic in Ω with gradient $\nabla\phi(z)$ continuous on $\Omega\cup\Gamma$, then given $\varepsilon>0$ there is a function $H(z)$ of the form given in (9) with

$$|\mathrm{Re}\, H(z) - \phi(z)| < \varepsilon \tag{10}$$

and

$$|\overline{H'(z)} - \nabla\phi(z)| < \varepsilon, \tag{11}$$

both holding for all z in $\Omega\cup\Gamma$.

One way to use this theorem to approximate the solution of (5) and (6) is to think of minimizing

$$\int_{\Gamma_1} (\mathrm{Re}\, H(z) - g_1(z))^2 |\mathrm{d}z| + \int_{\Gamma_2} (\overline{H'(z)}\cdot n(z) - g_2(z))^2 |\mathrm{d}z|. \tag{12}$$

Then the least-squares equations to determine the coefficients of $H(z)$ are obtained by setting to zero the partial derivatives of (12) with respect to the real and the imaginary parts of the coefficients in $H(z)$.

Another way to determine $H(z)$ is to choose $\{z_1,\ldots,z_{m_1}\}$ on Γ_1 and $\{z_1',\ldots,z_{m_2}'\}$ on Γ_2, and consider the discrete version of (5) and (6):

$$\mathrm{Re}\, H(z_j) = g_1(z_j) \quad \text{for } z_j \text{ on } \Gamma_1, \tag{13}$$

$$\overline{H'(z_j')\cdot n(z_j)} = g_2(z_j') \quad \text{for } z_j' \text{ on } \Gamma_2. \tag{14}$$

If these equations are overdetermined they can be solved in the least-squares sense.

The hypotheses that both $\phi(z)$ and $\nabla\phi(z)$ be continuous on Γ is physically clear for most applications, but oversimplification in modelling can create discontinuities. One such oversimplification is the formulation given in (5) and (6); a continuous version would write the boundary conditions in terms of

$$\alpha(z)\phi(z) + \beta(z)\nabla\phi(z), \tag{15}$$

where $\alpha(z)$ and $\beta(z)$ varied continuously on Γ, rather than with the jump discontinuities of (5) and (6). Examples are given in [8] showing that rather innocuous appearing mixed boundary value problems can have a solution which does not have $\nabla\phi(z)$ continuous on all of Γ. Some of these problems can be addressed by the use of integral norms which allow discontinuities in the boundary conditions.

A mixed boundary value problem on a domain Ω which has m holes has an approximate solution $H(z)$ satisfying (5) and (6) which has the generalized CVBEM form

$$H(z) = H_0(z) + H_1\left(\frac{1}{z-\alpha_1}\right) + \cdots + H_m\left(\frac{1}{z-\alpha_m}\right), \tag{16}$$

where each $H_j(z)$ has the form given in (9) and a_j is a point inside the jth hole. For details see [8].

The practical aspects of use of the CVBEM functions of this section to numerically solve mixed boundary value problems need to be studied.

5. A general CVBEM method

Consider two facts noted above. First, the nodes $\{\beta_j\}$ need not lie on Γ in order to have a CVBEM-type sum (2) give approximate solutions to Dirichlet problems. Second, sums as in (9) can be used instead of the sums in (2) for the numerical solution of Dirichlet problems, since that is a special case of a mixed boundary value problem. These facts raise the question of what functions can be used in CVBEM-type methods. The surprising answer is that practically anything will work, in the following sense [9].

Suppose that $f(z)$ is a function analytic on some neighborhood in the plane which is not a polynomial. The solution to any two-dimensional Dirichlet problem, with continuous data given on the boundary Γ of a bounded domain Ω with connected complement, can be approximated to an arbitrary degree of accuracy by sums of the form

$$\mathrm{Re}\sum_{n=1}^{N} c_n f(\alpha_n z + z_0), \tag{17}$$

with z_0, $\{c_n\}$ and $\{a_n\}$ complex numbers. The function $f(z)$ appearing in (17) could not be a polynomial (of degree m), since then sum (17) would be a polynomial of degree less than or equal to m and so could not approximate with arbitrary accuracy the solution to Dirichlet problem which had exact solution a polynomial of degree $m+1$. But, as the theorem states, anything else can be used!

It is a standard exercise [9] to show that for any analytic function $f(z)$ defined on a ball $B(z_1, r)$ which is not a polynomial, there is a point z_0 in this ball with the property that neither $f(z)$ nor any of its derivatives vanishes at z_0; the z_0 in (17) is taken to be such a point.

To establish the connection between sums (2) and (17), begin by considering the function $1/z$. Two integrations give $f(z) = -z + z\log(z)$, where $\log(z)$ has as branch cut the non-negative x-axis. For $z_0 = -1$ the functions

$$h_\alpha(z) = -(\alpha z - 1) + (\alpha x - 1)\log(\alpha z - 1) \tag{18}$$

are analytic on $B(0,1)$ for $|\alpha| < 1$.

Because

$$(z-\beta)\log_\beta(z-\beta) \tag{19}$$

and

$$\beta\left(\frac{1}{\beta}z - 1\right)\log\left(\frac{1}{\beta}z - 1\right) \tag{20}$$

have the same second derivative, it follows that on a common connected domain of analyticity these two functions differ by a linear function of z. If $\Omega \cup \Gamma$ is contained in $B(0,R)$, then for $|\beta| > R$, functions (19) and (20) are both analytic on $\Omega \cup \Gamma$. Combining all the linear terms which arise from the relations between these two function, sum (17) can be written as

$$\alpha_0 + \alpha_0' z + \sum_{n=1}^{N} \alpha_n \left(\frac{1}{\beta_n}z - 1\right)\log\left(\frac{1}{\beta_n}z - 1\right) \tag{21}$$

in direct correspondence with (2).

On aspect of the CVBEM sum (2) that does not follow from the theorem of this section is that the nodes $\{\beta_n\}$ in (2) can be chosen to lie on Γ, a result that depends on the specific structure of the logarithm and which requires curvilinear branch cuts if the domain is not convex. In numerical work nodes are often placed on Γ.

If a given Dirichlet problem has the property that the boundary function $g(z)$ is close to a second boundary function and for that second function the problem has a known solution $\mathrm{Re}f(z)$, then it seems likely that using that function f in the sums as given by (17) would give better numerical results than the standard sums (2). There has been no work done comparing how either numerical or theoretical properties of solutions vary with the choice of the function f.

6. The Dirichlet problem in higher dimensions

It obvious that real-world potential problems are generally three-dimensional, and only in very special cases can they be reduced to a problem in two dimensions, to which the CVBEM method as discussed so far be applied. Not only can the CVBEM method be extended to problems in higher dimensions, but this can be done using only analytic functions of one complex variable. This is a surprising result in light of classical examples, e.g. the Lebesgue spine or thorn [10], which exhibits a three-dimensional domain Ω and a continuous boundary function for which the corresponding Dirichlet problem has no solution, and yet on any two-dimensional section of Ω any Dirichlet problem with continuous boundary values can be solved.

Suppose that $h(x,y)$ is a harmonic function defined on the plane. How could it be used to construct a function harmonic on R^n? Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be points in R^n, taken with the usual dot product. Compute the Laplacian $\Delta H(\mathbf{x})$ of $H(\mathbf{x}) = h(\mathbf{a}\cdot\mathbf{x}, \mathbf{b}\cdot\mathbf{x})$ as

$$h_{11}(\mathbf{a}\cdot\mathbf{x}, \mathbf{b}\cdot\mathbf{x})\sum_1^n a_j^2 + h_{22}(\mathbf{a}\cdot\mathbf{x}, \mathbf{b}\cdot\mathbf{x})\sum_1^n b_j^2 + 2h_{12}(\mathbf{a}\cdot\mathbf{x}, \mathbf{b}\cdot\mathbf{x})\sum_1^n a_j b_j. \tag{22}$$

Observe that if the vectors $\mathbf{a}$ and $\mathbf{b}$ are chosen to have equal length and to be perpendicular, then $H(\mathbf{x})$ is harmonic in R^n. This simple example turns out to be the basis for extending the CVBEM to problems in R^n.

The most general conditions that a domain Ω in R^n must satisfy in order that every Dirichlet problem with continuous boundary data be solvable are complicated, but a condition which is sufficient for most applied problems is the Poincaré exterior cone condition, namely, that Ω be an open set in R^n with the property that for each point β in its boundary there is an open truncated cone K with vertex β and $K - \beta$ lying in $R^n - \Omega$.

The main theorem of [11] is:

Theorem 1. *Let $f(z)$ be a function analytic in a disk $D(z_0, \rho)$, where it is not a polynomial; we can suppose that neither f nor any of its derivatives vanish at the point z_0. Let Ω be a bounded domain in R^n, $n > 2$, with $R^n - \Omega$ connected and Ω satisfying the Poincaré exterior cone condition at each point of its boundary. Let $g(x)$ a real-valued function on the boundary Γ of Ω and $\varepsilon > 0$ be given. There are complex constants α_j and vectors $\mathbf{a}^j$ and $\mathbf{b}^j$ in R^n, $j = 1, 2, \ldots, N$, with*

$$\mathbf{a}^j \cdot \mathbf{b}^j = 0 \quad \text{and} \quad |\mathbf{a}^j| = |\mathbf{b}^j| < r, \tag{23}$$

r chosen with

$$|\mathbf{a}^j\cdot\mathbf{x} + \mathrm{i}\mathbf{b}^j\cdot\mathbf{x}| < \rho \quad \text{for all } \mathbf{x} \text{ in } \overline{\Omega}, \tag{24}$$

so that the corresponding function

$$h(\mathbf{x}) = \mathrm{Re}\sum_{j=1}^{N} \alpha_j f(z_0 + (\mathbf{a}^j\cdot\mathbf{x} + \mathrm{i}\,\mathbf{b}^j\cdot\mathbf{x})) \tag{25}$$

is defined and harmonic on $\overline{\Omega}$ and it satisfies

$$|h(\mathbf{x}) - g(\mathbf{x})| < \varepsilon \quad \text{for } \mathbf{x} \text{ in } \partial\Omega. \tag{26}$$

Consequently for all $\mathbf{x}$ in $\Omega \cup \Gamma$, $h(\mathbf{x})$ is within ε of the exact solution $u(\mathbf{x})$ to the Dirichlet problem with boundary data g.

Numerical applications of this theorem are found in [12–15]. Criteria for good choices for the vectors $\{\mathbf{a}^j\}$ and $\{\mathbf{b}^j\}$ would be useful. Nothing has been published concerning the numerical consequences of the choice of the function $f(z)$ in the theorem; most work has used the traditional functions given in Eq. (2). Some comparisons with other methods have been made but more needs to be done. The proof given is non-constructive, ultimately depending on the non-constructive proof of [9], so it cannot be applied to estimate the number N of terms in (25) required to achieve the desired accuracy. These results can no doubt be extended to mixed boundary value problems, and to domains with holes, but this has not been done.

The last theorem in [11] gives a particularly simple characterization of harmonic polynomials, the consequences of which have not been explored.

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