Approximating Solutions to the Dirichlet Problem in R^N Using One Analytic Function

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Received 6 February 2009; accepted 18 May 2009
Published online in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/num.20515

A simpler proof is given of the result of (Whitley and Hromadka II, Numer Methods Partial Differential Eq 21 (2005) 905–917) that, under very mild conditions, any solution to a Dirichlet problem with given continuous boundary data can be approximated by a sum involving a single function of one complex variable; any analytic function not a polynomial can be used. This can be applied to give a method for the numerical solution of potential problems in dimension three or higher. © 2009 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 000: 000–000, 2009

Keywords: complex variable methods; Dirichlet problem; harmonic polynomials

I. INTRODUCTION

A new method for the numerical solution of the Dirichlet problem was given in [1] in which it was shown that under mild conditions on a bounded open set Ω any solution to a Dirichlet problem with given continuous boundary data on $\partial\Omega$ can be approximated by a sum involving a single function of one complex variable; any analytic function not a polynomial can be used. This approximation has a form simple enough that it can be used in the numerical solution of Dirichlet problems in dimension three or higher. The proof given here, a substantial simplification of that given in [1], is obtained by proving the theorems of [1] in reverse order.

II. HARMONIC POLYNOMIALS

A complex-valued polynomial $P(x)$ on R^N can be written using the standard multi-index notation: $\alpha_1, \alpha_2, \ldots, \alpha_N$ non-negative integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, $x = (x_1, x_2, \ldots, x_N)$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$, as the finite sum $P(x) = \sum_\alpha C_\alpha x^\alpha$. This sum can be written as $P(x) = \sum P_m(x)$, where $P_m(x) = \sum_{|\alpha|=m} C_\alpha x^\alpha$, with $|\alpha| = \alpha_1 + \cdots + \alpha_N$, is a homogeneous polynomial of

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degree m, i.e., $P_m(\lambda x) = \lambda^m P_m(x)$ for λ real. In this way of writing $P(x)$, the polynomials $P_m(x)$ are uniquely determined and $P(x)$ is harmonic if and only if each $P_m(x)$ is harmonic, all of which follows directly upon substituting λx for x and considering the resulting polynomial in λ [2, pp. 23–24]. Consequently, in studying harmonic polynomials on R^N it generally suffices to consider the harmonic polynomials on R^N which are homogeneous of degree m, denoted by $\mathcal{H}_m(R^N)$.

As noted in [3, 2.1], and [2, pp. 81–82], the space $\mathcal{H}_m(R^2)$ has a "very simple form" in that it has a basis consisting of the functions $Re(x_1 + i\ x_2)^m$ and $Im(x_1 + i\ x_2)^m$, which using complex-valued functions can be written

$$\mathcal{H}_m(R^2) = sp\{(x_1 + ix_2)^m, (x_1 - ix_2)^m)\}$$

The purpose of this section is to establish a similar representation for general N. The following elementary lemma [1] is useful.

Lemma 1. *Let h be an harmonic function of two variables defined on an open set U in R^2, and let a and b be two perpendicular vectors, $a \cdot b = 0$, of equal length $|a| = |b|$ in R^N. Then $h(a \cdot x, b \cdot x)$ is an harmonic function for x in R^N and $(a \cdot x, b \cdot x)$ in U.*

Proof. Set $H(x) = h(a \cdot x, b \cdot x)$. The result follows from the computation of the Laplacian $\Delta H(x)$:

$$h_{11}(a \cdot x, b \cdot x)\sum_1^N a_j^2 + h_{22}(a \cdot x, b \cdot x)\sum_1^N b_j^2 + 2h_{12}(a \cdot x, b \cdot x)\sum_1^N a_j b_j. \tag{1}$$

■

Let the set A_N consist of the pairs a,b of orthogonal points in R^N, $a \cdot b = 0$, having unit length, $|a| = |b| = 1$, and define the complex vector space spanned by functions on R^N of the form $(a \cdot x + i\ b \cdot x)^m$:

$$\mathcal{A}_m^N = sp\{(a \cdot x + i\ b \cdot x)^m : \text{for pairs } a, b \text{ in } A_N\}$$

Each function in $\mathcal{A}_m^N$ is harmonic by Lemma 1.

The choice of whether to consider real-valued harmonic functions [3] or complex-valued harmonic functions [2] is merely a question of which notation is more convenient, like the choice of considering Fourier series as series in $\{\sin(n\theta), \cos(n\theta)\}$ or as a series in $\{e^{in\theta}\}$. For $\mathcal{A}_m^N$, the choice of complex-valued harmonic functions gives the simpler notation, keeping in mind that if the pair a, b belongs to A_N, then so does the pair $a, -b$, i.e., both $(a \cdot x + ib \cdot x)^m$ and $(a \cdot x - ib \cdot x)^m$, and therefore $Re(a \cdot x + ib \cdot x)^m$ and $Im(a \cdot x + ib \cdot x)^m$ are in $\mathcal{A}_m^N$

As $\mathcal{H}_m(R^N)$ is finite-dimensional, all norms on it are equivalent; below the supremum norm will be used:

$$\|p(x)\| = \sup\{|p(x)| : |x| \leq 1\}.$$

For an harmonic function, the maximum principle implies that the supremum can be restricted to x on the surface of the unit sphere in R^N.

Lemma 2. $\mathcal{A}_m^N = sp\{(a \cdot x + i\ b \cdot x)^m$*: for pairs a,b in A_N with $a_1 + ib_1 \neq 0\}$*

Consider $(a \cdot x + ib \cdot x)^m$ in $\mathcal{A}_m^N$ in the case where $a_1 + ib_1 = 0$. Define, for each $0 < \delta < 1$, $a(\delta)$ in R^N by

$$a(\delta) = (\delta, \sqrt{1-\delta^2}a_2, \ldots, \sqrt{1-\delta^2}a_N). \tag{2}$$

As $|a(\delta)| = 1 = |b|$ and $a(\delta) \cdot b = 0$, the pair $a(\delta), b$ belongs to A_N and therefore $(a(\delta) \cdot x + ib \cdot x)^m$ belongs to

$$L = sp\{(a \cdot x + i\ b \cdot x)^m : \text{for pairs } a, b \text{ in } A_N \text{ with } a_1 + ib_1 \neq 0\},$$

a subspace of $\mathcal{A}_m^N$. As $|a(\delta) - a|$ converges to zero as $\delta \to 0$,

$$\|(a(\delta) \cdot x + ib \cdot x)^m - (a \cdot x + ib \cdot x)^m\| \to 0.$$

Thus for a finite sum $p(x) = \sum c_j (a^{(j)} \cdot x + ib^{(j)} \cdot x)^m$ in $\mathcal{A}_m^N$, if in each term where $a_1^{(j)} + b_1^{(j)} = 0$ the element a is modified as in (2), the resulting modified sum $\hat{p}(x)$ is in L and $\|p - \hat{p}\|$ can be made less than any preassigned $\epsilon > 0$ for δ chosen small enough. This shows that L is dense in $\mathcal{A}_m^N$, but L being finite dimensional is closed so it must equal $\mathcal{A}_m^N$. ■

Theorem 1. *For all m and N,*

$$\mathcal{H}_m(R^N) = \mathcal{A}_m^N. \tag{3}$$

Proof. The proof will be by induction on the dimension $N \geq 2$, and then a further induction on those m for which the statement (3) holds for the N under consideration.

It has been noted that (3) holds for $N = 2$ and all $m = 0, 1, \ldots$. For any N, (3) is obviously true for $m = 0$; for $m = 1$, the corresponding homogeneous polynomials of degree one are given by $x_j = [(x_j + ix_k) + (x_j + i(-1)x_k)]/2$ for $k \neq j$.

To start the induction, suppose that (3) holds for some N and for that N, for all m. Consider N+1, and as (3) holds for $m = 0$ and $m = 1$, it will be supposed that it holds for some m and it will be shown then that (3) holds for m+1.

Let u be a function in $\mathcal{H}_{m+1}(R^{N+1})$ The partial derivative of u with respect to the first variable x_1, $D_1 u$, is a harmonic polynomial homogeneous of degree m, and by the induction hypothesis can be written as the finite sum

$$D_1 u = \sum c_j (a^{(j)} \cdot x + i\ b^{(j)} \cdot x)^m, \tag{4}$$

c_j complex constants and each pair $a^{(j)}, b^{(j)}$ belonging to A_{N+1}. Applying Lemma 2, it can be further be assumed that $a_1^{(j)} + ib_1^{(j)} \neq 0$ for each j. Define

$$v = \sum c_j' (a^{(j)} \cdot x + i\ b^{(j)} \cdot x)^{m+1}, \tag{5}$$

with

$$c_j' = \frac{c_j}{(m+1)(a_1^{(j)} + i\ b_1^{(j)})}.$$

Then v belongs to $\mathcal{A}_{m+1}^{N+1}$ and $u - v$ belongs to $\mathcal{H}_{m+1}(R^{N+1})$ with $D_1(u - v)$ zero. Write $u - v = \sum_{|\alpha|=m+1} C_\alpha x^\alpha$, then $D_1(u - v) = \sum_{|\alpha|=m+1} \alpha_1 C_\alpha x^{\alpha - e_1}$, $e_1 = (1, 0, \ldots, 0)$, showing that $u - v$ is an harmonic polynomial in the variables $x_2, x_3, \ldots, x_{N+1}$, having the form

$$u - v = \sum C_\alpha x^\alpha = \sum C_{(0,\alpha_2,\ldots,\alpha_{N+1})} x_2^{\alpha_2} \ldots x_{N+1}^{\alpha_{N+1}}, \tag{6}$$

the sum taken over all $\alpha_2 + \cdots + \alpha_{N+1} = m + 1$. This makes it clear that $u - v$ is an harmonic function, homogeneous of degree $m + 1$, in the N- variables $x_2, \ldots, x_{N+1}$, and as such by the induction hypothesis can be written as a linear combination of the functions of the form

$$[(a_2, \ldots, a_{N+1}) \cdot (x_2, \ldots x_{N+1}) + i\ (b_2, \ldots, b_{N+1}) \cdot (x_2, \ldots x_{N+1})]^{m+1},$$

the pair $(a_2, \ldots, a_{N+1})$, $(b_2, \ldots, b_{N+1})$ belonging to A_N. Each of these terms can be written

$$[(0, a_2, \ldots, a_{N+1}) \cdot (x_1, \ldots, x_{N+1}) + i\ (0, b_2, \ldots, b_{N+1}) \cdot (x_1, x_2, \ldots, x_{N+1})]^{m+1},$$

the pairs $(0, a_2, \ldots, a_{N+1})$, $(0, b_2, \ldots, b_{N+1})$ belonging to A_{N+1}. Thus the sum representing $u - v$ belongs to $\mathcal{A}_{m+1}^{N+1}$, as does v, and therefore so does u. ∎

III. THE DIRICHLET PROBLEM

The basic Dirichlet problem for a domain Ω in R^N is: Given a function g defined and continuous on the boundary $\partial\Omega$ of Ω, find a function u harmonic in Ω and continuous on the closure $\overline{\Omega}$ with $u = g$ on the boundary.

Lemma 3. *Let f be analytic on the disc $D(z_1, r) = \{z : |z - z_1| < r\}$. If f is not a polynomial, there is a point z_0 in this disk where f and every derivative of f is not zero:*

$$f^{(n)}(z_0) \neq 0 \text{ for } n = 0, 1, \ldots. \tag{7}$$

Proof. See [4, ex. 2, p. 227] or [5]. Let $D_n = \{z : f^{(n)}(z) = 0\}$. If the lemma is false, $D(z_1, r) \subset \cup(D_n)$ and any closed uncountable subset F of $D(z_1, r)$ intersects at least one D_n in an infinite set with a limit point in F; by the identity theorem $f^{(n)}$ is identically zero in $D(z_1, r)$ and f is a polynomial. ∎

With reference to the above lemma, a linear change of variable applied to any function analytic and not a polynomial on some disk will give a function satisfying the conditions on the function f in Theorem 2 below.

Theorem 2. *Let Ω be a bounded open subset of R^N, with the property that given any continuous function g on its boundary, for each $\epsilon > 0$, there is a harmonic polynomial p with $|p(x) - g(x)| < \epsilon$ for all x in $\partial\Omega$.*

Let f be analytic in the disk $D(0, r)$ which contains Ω, and further suppose that $f^{(j)}(0) \neq 0$, for $j = 0, 1, \ldots$. Let a continuous function g be given on $\partial\Omega$. For any $\epsilon > 0$, there are a finite number of pairs of elements a^k, b^k in A_N, λ_k real, $|\lambda_k| \leq 1/4$ and complex coefficients c_k, with

$$|g(x) - \sum c_k f(\lambda_k(a^k \cdot x + ib^k \cdot x))| \leq \epsilon \text{ for all } x \text{ in } \partial\Omega. \tag{8}$$

Proof. Consider the Banach space $C(\partial\Omega)$ of all continuous (complex- valued) functions defined on $\partial\Omega$, taken with the supremum norm. The theorem states that the subspace M spanned by all sums of the form given in (8) is dense in $C(\partial\Omega)$. If this is not so, there is a function g in $C(\partial\Omega)$ not in the closure of M. By the Hahn-Banach theorem, there is a continuous linear functional x^* which is zero on M and has $x^*(g) \neq 0$.

As x^* annihilates the subspace M, it must annihilate $f(\lambda(a \cdot x + ib \cdot x))$ for all a, b in A_N and all λ, $|\lambda| \leq 1/4$. On the closure of the the ball $B(0, r/2)$ the power series $f(z) = \sum c'_j z^j$ for f converges uniformly. By the continuity of x^*,

$$0 = x^*[f(\lambda(a \cdot x + ib \cdot x))] = \sum c'_j \lambda^j x^*((a \cdot x + ib \cdot x)^j) \tag{9}$$

for all λ and a, b of the prescribed type.

As none of the coefficients c'_j are zero, by regarding the series (9) as a power series in λ (clearly a smaller set of λ than all those satisfying $|\lambda| \leq r/4$ will suffice) it is seen that $x^*((a \cdot x + ib \cdot x)^j) = 0$ for all j and all pairs a, b in A_N. From Theorem 1, $x^*(p_j) = 0$ for all harmonic polynomials p_j, homogeneous of degree j, and hence $x^*(p) = 0$ for all harmonic polynomials. By hypothesis, such polynomials are dense in $C(\partial\Omega)$ and x^* is zero, contrary to assumption.

The Dirichlet problem is solvable for any continuous boundary function g for a domain with the property hypothesized in the theorem. For if $p^{(k)}$ are harmonic polynomials with $|g(x) - p^{(k)}(x)|$ converging to zero uniformly on $\partial\Omega$, the sequence of polynomials $\{p^{(k)}\}$ is Cauchy in $C(\partial\Omega)$ and so converges uniformly on $\partial\Omega$ to g, and by the maximum principle uniformly on the closure of Ω to a function u which is harmonic in Ω [2, p. 16], continuous on the closure, and equal to g on the boundary. The maximum principle implies that two harmonic functions which are close on the boundary of Ω are also close throughout Ω, and so u is approximated by the sum in (8) throughout the closure of Ω. ■

If Ω is a bounded open set of R^N, with $R^N - \overline{\Omega}$ connected, the condition above on Ω, that any Dirichlet problem with continuous boundary data has a solution which can be approximated by an harmonic polynomial, is equivalent to the condition that $R^N - \overline{\Omega}$ is not thin at each point of $\partial\Omega$, see [6, Theorem 1.15], [3, Theorem 7.9.7], [1, Theorem 1], and the proof of this is more technically difficult than the results proved in this note. A condition that suffices for applications is that the domain has the hypothesized property of Theorem 2 if it satisfies the Poincare exterior cone condition: at each point ζ in the boundary of Ω, there is an open truncated cone C with vertex ζ and $C - \{\zeta\}$ lying in $R^N - \Omega$ [2, Chapter 11], [3, Chapter 6].

Theorem 2 gives a method for the numerical solution of the Dirichlet problem in R^3 (or R^N) which is particularly simple when using a programming language with a complex data type and built-in subroutines for some analytic functions. One chooses an analytic function f, some points in A_N, and fits a sum of the type described in the Theorem, say by least squares, to the given function g on $\partial\Omega$. See [7] for references to numerical results and a discussion of prior work.

References

1. R. Whitley and T. Hromadka II, Approximating harmonic functions on R^n with one function of a single complex variable, Numer Methods Partial Differential Eq 21 (2005), 905–917.
2. S. Axler, P. Bourdon, and W. Ramey, Harmonic function theory, 2nd edition, Springer-Verlag, New York, 2001.
3. D. Armitage and S. Gardiner, Classical potential theory, Springer-Verlag, London, 2001.

4. W. Rudin, Real and complex analysis, 3rd Ed., McGraw-Hill, New York, 1987.
5. R. Whitley and T. Hromadka II, A general complex variable boundary element method, Numer Methods Partial Differential Eq 17 (2001), 332–335.
6. S. J. Gardiner, Harmonic Approximation, London Mathematical Society, Lecture Note Series 221, Cambridge University Press, Cambridge, UK, 1995.
7. R. Whitley and T. Hromadka II, Theoretical developments in the complex boundary element method, Eng Anal Boundary Elements 30 (2006), 1020–1024.