Approximating Solutions to the Dirichlet Problem in $R^N$ Using One Analytic Function

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A simpler proof is given of the result of (Whitley and Hromadka II, Numer Methods Partial Differential Eq 21 (2005) 905–917) that, under very mild conditions, any solution to a Dirichlet problem with given continuous boundary data can be approximated by a sum involving a single function of one complex variable; any analytic function not a polynomial can be used. This can be applied to give a method for the numerical solution of potential problems in dimension three or higher. © 2009 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 000: 000–000, 2009

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I. INTRODUCTION

A new method for the numerical solution of the Dirichlet problem was given in [1] in which it was shown that under mild conditions on a bounded open set $\Omega$ any solution to a Dirichlet problem with given continuous boundary data on $\partial \Omega$ can be approximated by a sum involving a single function of one complex variable; any analytic function not a polynomial can be used. This approximation has a form simple enough that it can be used in the numerical solution of Dirichlet problems in dimension three or higher. The proof given here, a substantial simplification of that given in [1], is obtained by proving the theorems of [1] in reverse order.

II. HARMONIC POLYNOMIALS

A complex-valued polynomial $P(x)$ on $R^N$ can be written using the standard multi-index notation: $\alpha_1, \alpha_2, \ldots, \alpha_N$ non-negative integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, $x = (x_1, x_2, \ldots, x_N)$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$, as the finite sum $P(x) = \sum_{\alpha} C_\alpha x^\alpha$. This sum can be written as $P(x) = \sum P_m(x)$, where $P_m(x) = \sum_{|\alpha| = m} C_\alpha x^\alpha$, with $|\alpha| = \alpha_1 + \cdots + \alpha_N$, is a homogeneous polynomial of
degree \( m \), i.e., \( P_m(\lambda x) = \lambda^m P_m(x) \) for \( \lambda \) real. In this way of writing \( P(x) \), the polynomials \( P_m(x) \) are uniquely determined and \( P(x) \) is harmonic if and only if each \( P_m(x) \) is harmonic, all of which follows directly upon substituting \( \lambda x \) for \( x \) and considering the resulting polynomial in \( \lambda \) [2, pp. 23–24]. Consequently, in studying harmonic polynomials on \( \mathbb{R}^N \) it generally suffices to consider the harmonic polynomials on \( \mathbb{R}^N \) which are homogeneous of degree \( m \), denoted by \( \mathcal{H}_m(\mathbb{R}^N) \).

As noted in [3, 2.1], and [2, pp. 81–82], the space \( \mathcal{H}_m(\mathbb{R}^2) \) has a “very simple form” in that it has a basis consisting of the functions \( \Re(x_1 + ix_2)^m \) and \( \Im(x_1 + ix_2)^m \), which using complex-valued functions can be written

\[
\mathcal{H}_m(\mathbb{R}^2) = \text{span}\{(x_1 + ix_2)^m, (x_1 - ix_2)^m\}
\]

The purpose of this section is to establish a similar representation for general \( N \). The following elementary lemma [1] is useful.

**Lemma 1.** Let \( h \) be an harmonic function of two variables defined on an open set \( U \) in \( \mathbb{R}^2 \), and let \( a \) and \( b \) be two perpendicular vectors, \( a \cdot b = 0 \), of equal length \( |a| = |b| \) in \( \mathbb{R}^N \). Then \( h(a \cdot x, b \cdot x) \) is an harmonic function for \( x \) in \( \mathbb{R}^N \) and \( (a \cdot x, b \cdot x) \) in \( U \).

**Proof.** Set \( H(x) = h(a \cdot x, b \cdot x) \). The result follows from the computation of the Laplacian

\[
\Delta H(x) = h_{11}(a \cdot x, b \cdot x) \sum_{1}^{N} a_j^2 + h_{22}(a \cdot x, b \cdot x) \sum_{1}^{N} b_j^2 + 2h_{12}(a \cdot x, b \cdot x) \sum_{1}^{N} a_j b_j.
\]

\[ (1) \]

Let the set \( A_N \) consist of the pairs \( a, b \) of orthogonal points in \( \mathbb{R}^N \), \( a \cdot b = 0 \), having unit length, \( |a| = |b| = 1 \), and define the complex vector space spanned by functions on \( \mathbb{R}^N \) of the form \( (a \cdot x + ib \cdot x)^m \):

\[
\mathcal{A}_m^N = \text{span}\{(a \cdot x + ib \cdot x)^m : \text{for pairs } a, b \text{ in } A_N\}
\]

Each function in \( \mathcal{A}_m^N \) is harmonic by Lemma 1.

The choice of whether to consider real-valued harmonic functions [3] or complex-valued harmonic functions [2] is merely a question of which notation is more convenient, like the choice of considering Fourier series as series in \( \{\sin(n\theta), \cos(n\theta)\} \) or as a series in \( \{e^{in\theta}\} \). For \( \mathcal{A}_m^N \), the choice of complex-valued harmonic functions gives the simpler notation, keeping in mind that if the pair \( a, b \) belongs to \( A_N \), then so does the pair \( a, -b \), i.e., both \( (a \cdot x + ib \cdot x)^m \) and \( (a \cdot x - ib \cdot x)^m \), and therefore \( \Re(a \cdot x + ib \cdot x)^m \) and \( \Im(a \cdot x + ib \cdot x)^m \) are in \( \mathcal{A}_m^N \).

As \( \mathcal{H}_m(\mathbb{R}^N) \) is finite-dimensional, all norms on it are equivalent; below the supremum norm will be used:

\[
\|p(x)\| = \sup\{|p(x)| : |x| \leq 1\}.
\]

For an harmonic function, the maximum principle implies that the supremum can be restricted to \( x \) on the surface of the unit sphere in \( \mathbb{R}^N \).

**Lemma 2.** \( \mathcal{A}_m^N = \text{span}\{(a \cdot x + ib \cdot x)^m : \text{for pairs } a, b \text{ in } A_N \text{ with } a_1 + ib_1 \neq 0\} \)
Consider \((a \cdot x + ib \cdot x)^m\) in \(A^N_m\) in the case where \(a_1 + ib_1 = 0\). Define, for each \(0 < \delta < 1\), \(a(\delta)\) in \(R^N\) by
\[
a(\delta) = (\delta, \sqrt{1 - \delta^2}a_2, \ldots, \sqrt{1 - \delta^2}a_N).
\]

As \(|a(\delta)| = 1 = |b|\) and \(a(\delta) \cdot b = 0\), the pair \(a(\delta), b\) belongs to \(A_N\) and therefore \((a(\delta) \cdot x + ib \cdot x)^m\) belongs to
\[L = sp\{(a \cdot x + i b \cdot x)^m : \text{for pairs } a, b \in A_N \text{ with } a_1 + ib_1 \neq 0\},\]
a subspace of \(A^N_m\). As \(|a(\delta) - a|\) converges to zero as \(\delta \to 0\),
\[
\|a(\delta) \cdot x + ib \cdot x)^m - (a \cdot x + i b \cdot x)^m\| \to 0.
\]
Thus for a finite sum \(p(x) = \sum c_j (a^{(j)} \cdot x + ib^{(j)} \cdot x)^m\) in \(A^N_m\), if in each term where \(a^{(j)}_1 + b^{(j)}_1 \neq 0\) the element \(a\) is modified as in (2), the resulting modified sum \(\tilde{p}(x)\) is in \(L\) and \(\|p - \tilde{p}\|\) can be made less than any preassigned \(\epsilon > 0\) for \(\delta\) chosen small enough. This shows that \(L\) is dense in \(A^N_m\), but \(L\) being finite dimensional is closed so it must equal \(A^N_m\).

**Theorem 1.** For all \(m\) and \(N\),
\[
\mathcal{H}_m(R^N) = A^N_m. \quad (3)
\]

**Proof.** The proof will be by induction on the dimension \(N \geq 2\), and then a further induction on those \(m\) for which the statement (3) holds for the \(N\) under consideration.

It has been noted that (3) holds for \(N = 2\) and all \(m = 0, 1, \ldots\). For any \(N\), (3) is obviously true for \(m = 0\); for \(m = 1\), the corresponding homogeneous polynomials of degree one are given by \(x_j = [(x_j + ik_x) + (x_j + i(-1)x_k)]/2\) for \(k \neq j\).

To start the induction, suppose that (3) holds for some \(N\) and for that \(N\), for all \(m\). Consider \(N+1\), and as (3) holds for \(m = 0\) and \(m = 1\), it will be supposed that it holds for some \(m\) and it will be shown then that (3) holds for \(m+1\).

Let \(u\) be a function in \(\mathcal{H}_{m+1}(R^{N+1})\). The partial derivative of \(u\) with respect to the first variable \(x_1\), \(D_1 u\), is a harmonic polynomial homogeneous of degree \(m\), and by the induction hypothesis can be written as the finite sum
\[
D_1 u = \sum c_j (a^{(j)} \cdot x + i b^{(j)} \cdot x)^m, \quad (4)
\]

\(c_j\) complex constants and each pair \(a^{(j)}, b^{(j)}\) belonging to \(A_{N+1}\). Applying Lemma 2, it can be further be assumed that \(a^{(j)}_1 + ib^{(j)}_1 \neq 0\) for each \(j\). Define
\[
v = \sum c'_j (a^{(j)} \cdot x + i b^{(j)} \cdot x)^{m+1}, \quad (5)
\]

with
\[
c'_j = \frac{c_j}{(m+1)(a^{(j)}_1 + i b^{(j)}_1)}.
\]
Then $v$ belongs to $A_{m+1}^N$ and $u - v$ belongs to $H_{m+1}(R^{N+1})$ with $D_1(u - v)$ zero. Write $u - v = \sum |\alpha|=m+1 C_{\alpha} x^\alpha$, then $D_1(u - v) = \sum |\alpha|=m+1 \alpha_1 C_{\alpha} x^{\alpha_1 - \epsilon_1}$, $\epsilon_1 = (1, 0, \ldots, 0)$, showing that $u - v$ is an harmonic polynomial in the variables $x_2, x_3, \ldots, x_{N+1}$, having the form

$$u - v = \sum C_{\alpha} x^\alpha = \sum C_{(\alpha_2,\ldots,\alpha_{N+1})} x^{\alpha_2 \cdot \ldots \cdot x^{\alpha_{N+1}},}$$ (6)

the sum taken over all $\alpha_2 + \cdots + \alpha_{N+1} = m + 1$. This makes it clear that $u - v$ is an harmonic function, homogeneous of degree $m + 1$, in the $N$ variables $x_2, \ldots, x_{N+1}$, and as such by the induction hypothesis can be written as a linear combination of the functions of the form

$$[(a_2, \ldots, a_{N+1}) \cdot (x_2, \ldots, x_{N+1}) + i (b_2, \ldots, b_{N+1}) \cdot (x_2, \ldots, x_{N+1})]^m,$$

the pair $(a_2, \ldots, a_{N+1}), (b_2, \ldots, b_{N+1})$ belonging to $A_{N}$. Each of these terms can be written

$$[(0, a_2, \ldots, a_{N+1}) \cdot (x_1, \ldots, x_{N+1}) + i (0, b_2, \ldots, b_{N+1}) \cdot (x_1, x_2, \ldots, x_{N+1})]^m,$$

the pairs $(0, a_2, \ldots, a_{N+1}), (0, b_2, \ldots, b_{N+1})$ belonging to $A_{N+1}$. Thus the sum representing $u - v$ belongs to $A_{m+1}^N$, as does $v$, and therefore so does $u$. 

III. THE DIRICHLET PROBLEM

The basic Dirichlet problem for a domain $\Omega$ in $R^N$ is: Given a function $g$ defined and continuous on the boundary $\partial \Omega$ of $\Omega$, find a function $u$ harmonic in $\Omega$ and continuous on the closure $\overline{\Omega}$ with $u = g$ on the boundary.

Lemma 3. Let $f$ be analytic on the disc $D(z_1, r) = \{ z : |z - z_1| < r \}$. If $f$ is not a polynomial, there is a point $z_0$ in this disk where $f$ and every derivative of $f$ is not zero:

$$f^{(n)}(z_0) \neq 0 \text{ for } n = 0, 1, \ldots. \quad \text{(7)}$$

Proof. See [4, ex. 2, p. 227] or [5]. Let $D_n = \{ z : f^{(n)}(z) = 0 \}$. If the lemma is false, $D(z_1, r) \subset \cup D_n$ and any closed uncountable subset $F$ of $D(z_1, r)$ intersects at least one $D_n$ in an infinite set with a limit point in $F$; by the identity theorem $f^{(n)}$ is identically zero in $D(z_1, r)$ and $f$ is a polynomial.

With reference to the above lemma, a linear change of variable applied to any function analytic and not a polynomial on some disk will give a function satisfying the conditions on the function $f$ in Theorem 2 below.

Theorem 2. Let $\Omega$ be a bounded open subset of $R^N$, with the property that given any continuous function $g$ on its boundary, for each $\epsilon > 0$, there is a harmonic polynomial $p$ with $|p(x) - g(x)| < \epsilon$ for all $x$ in $\partial \Omega$.

Let $f$ be analytic in the disk $D(0, r)$ which contains $\Omega$, and further suppose that $f^{(j)}(0) \neq 0$, for $j = 0, 1, \ldots$. Let a continuous function $g$ be given on $\partial \Omega$. For any $\epsilon > 0$, there are a finite number of pairs of elements $a^k, b^k$ in $A_{\lambda_k}$, $\lambda_k$ real, $|\lambda_k| \leq 1/4$ and complex coefficients $c_k$, with

$$|g(x) - \sum c_k f(\lambda_k (a^k \cdot x + i b^k \cdot x))| \leq \epsilon \text{ for all } x \in \partial \Omega. \quad \text{(8)}$$

Proof. Consider the Banach space \( C(\partial \Omega) \) of all continuous (complex-valued) functions defined on \( \partial \Omega \), taken with the supremum norm. The theorem states that the subspace \( M \) spanned by all sums of the form given in (8) is dense in \( C(\partial \Omega) \). If this is not so, there is a function \( g \) in \( C(\partial \Omega) \) not in the closure of \( M \). By the Hahn-Banach theorem, there is a continuous linear functional \( x^* \) which is zero on \( M \) and has \( x^*(g) \neq 0 \).

As \( x^* \) annihilates the subspace \( M \), it must annihilate \( f(\lambda(a \cdot x + ib \cdot x)) \) for all \( a, b \) in \( A_N \) and all \( \lambda, |\lambda| \leq 1/4 \). On the closure of the ball \( B(0, r/2) \) the power series \( f(z) = \sum c_j z^j \) for \( f \) converges uniformly. By the continuity of \( x^* \),

\[
0 = x^* [ f(\lambda(a \cdot x + ib \cdot x))] = \sum c_j \lambda^j x^*(((a \cdot x + ib \cdot x)^j)
\]

for all \( \lambda \) and \( a, b \) of the prescribed type.

As none of the coefficients \( c_j \) are zero, by regarding the series (9) as a power series in \( \lambda \) (clearly a smaller set of \( \lambda \) than all those satisfying \( |\lambda| \leq r/4 \) will suffice) it is seen that \( x^*((a \cdot x + ib \cdot x)^j) = 0 \) for all \( j \) and all pairs \( a, b \) in \( A_N \). From Theorem 1, \( x^*(p_j) = 0 \) for all harmonic polynomials \( p_j \), homogeneous of degree \( j \), and hence \( x^*(p) = 0 \) for all harmonic polynomials. By hypothesis, such polynomials are dense in \( C(\partial \Omega) \) and \( x^* \) is zero, contrary to assumption.

The Dirichlet problem is solvable for any continuous boundary function \( g \) for a domain with the property hypothesized in the theorem. For if \( p^{(k)} \) are harmonic polynomials with \( |g(x) - p^{(k)}(x)| \) converging to zero uniformly on \( \partial \Omega \), the sequence of polynomials \( \{p^{(k)}\} \) is Cauchy in \( C(\partial \Omega) \) and so converges uniformly on \( \partial \Omega \) to \( g \), and by the maximum principle uniformly on the closure of \( \Omega \) to a function \( u \) which is harmonic in \( \Omega \) [2, p. 16], continuous on the closure, and equal to \( g \) on the boundary. The maximum principle implies that two harmonic functions which are close on the boundary of \( \Omega \) are also close throughout \( \Omega \), and so \( u \) is approximated by the sum in (8) throughout the closure of \( \Omega \).

If \( \Omega \) is a bounded open set of \( \mathbb{R}^N \), with \( \mathbb{R}^N - \overline{\Omega} \) connected, the condition above on \( \Omega \), that any Dirichlet problem with continuous boundary data has a solution which can be approximated by an harmonic polynomial, is equivalent to the condition that \( \mathbb{R}^N - \overline{\Omega} \) is not thin at each point of \( \partial \Omega \), see [6, Theorem 1.15], [3, Theorem 7.9.7], [1, Theorem 1], and the proof of this is more technically difficult than the results proved in this note. A condition that suffices for applications is that the domain has the hypothesized property of Theorem 2 if it satisfies the Poincare exterior cone condition: at each point \( \xi \) in the boundary of \( \Omega \), there is an open truncated cone \( C \) with vertex \( \xi \) and \( C - \{\xi\} \) lying in \( \mathbb{R}^N - \Omega \) [2, Chapter 11], [3, Chapter 6].

Theorem 2 gives a method for the numerical solution of the Dirichlet problem in \( \mathbb{R}^3 \) (or \( \mathbb{R}^N \)) which is particularly simple when using a programming language with a complex data type and built-in subroutines for some analytic functions. One chooses an analytic function \( f \), some points in \( A_N \), and fits a sum of the type described in the Theorem, say by least squares, to the given function \( g \) on \( \partial \Omega \). See [7] for references to numerical results and a discussion of prior work.

References

1. R. Whitley and T. Hromadka II, Approximating harmonic functions on \( \mathbb{R}^n \) with one function of a single complex variable, Numer Methods Partial Differential Eq 21 (2005), 905–917.

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