Section 17.1
Vector Fields

"Vector Fields, or Vector Valued Functions"

In this chapter we consider a new type of integral where instead of integrating a scalar valued function, we consider integrating vector valued functions. Though the definitions are very different, we shall see that the new integrals we shall define can be interpreted as regular double and triple integrals and evaluated using generalizations of the Fundamental Theorem of Calculus.

1. Vector Fields

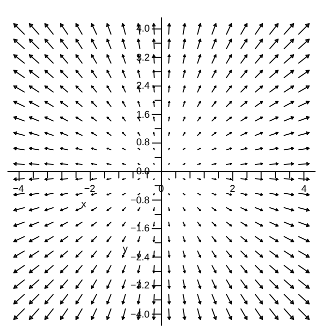
We start with a formal definition of a vector field.

Definition 1.1. A vector field is a function $\vec{F}$ on $\mathbb{R}^2$ (or $\mathbb{R}^3$) which assigns to each point in $\mathbb{R}^2$ (or $\mathbb{R}^3$) a two dimensional vector $\vec{F}(x,y)$ in 2-space (or a vector $\vec{F}(x,y,z)$ in 3-space).

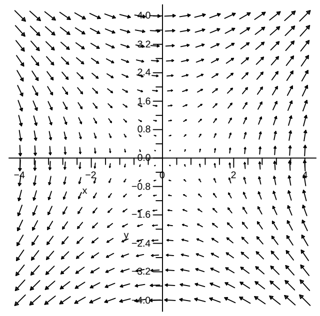
Since vectors can be decomposed, we can also decompose vector fields. Specifically, we can write $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ for a 2 dimensional vector field where P and Q are functions of two variables, and we can write $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ for a 3 dimensional vector field where P, Q and R are functions of three variables. We call P, Q and R the component functions of $\vec{F}$.
We can think of a vector field as an assignment to each point in the region we are considering a vector, or arrow. This gives us a way to illustrate vector fields. We illustrate with some examples.

Example 1.2. Sketch the following vector fields.

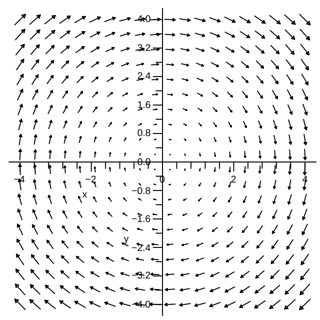
(*i*) $\vec{F} = x\vec{i} + y\vec{j}$ This is sometimes called the radial vector fields - all vectors point away from the origin.



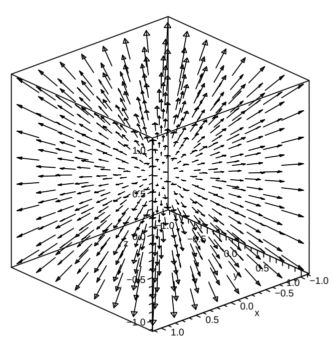
(*ii*) $\vec{F} = y\vec{i} + x\vec{j}$ We can sketch it by looking at a small number of vectors to get a general idea.



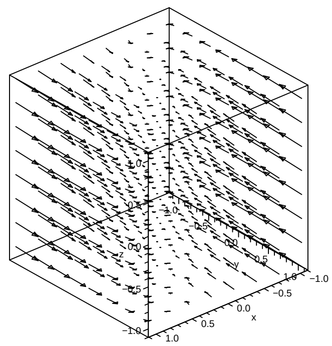
(*iii*) $\vec{F} = y\vec{i} - x\vec{j}$ We can sketch it by looking at a small number of vectors to get a general idea.



(*iv*) $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ This is sometimes called the radial vector fields - all vectors point away from the origin.



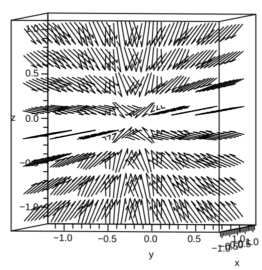
(*v*) $\vec{F} = y\vec{i} + 2x\vec{j} + (x+y)\vec{k}$ Observe that for this vector field, the vectors are independent of z-value (since no z's appear in any of the component functions). This means that the vectors will be the same for fixed x and y values as we move up and down.



(vi)

$$\vec{F} = \frac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}$$

This is like a gravitational force pulling all points towards the centre. Observe that since we are dividing by $\sqrt{x^2+y^2+z^2}$, each vector is a unit vector.



Observe that vector fields arise in many different disciplines (like physics - magnetics, gravitational fields, fluid dynamics; meteorology - weather predictions, wind, and so on).

2. Gradient Vector Fields

We have already seen an example of a vector field - namely the gradient vector of a function. We recall the definition.

Definition 2.1. If $f(x,y)$ is a differentiable function, then $\nabla f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$ is a vector field called the gradient vector field for f and we call f the potential function of ∇f.
Likewise, if $f(x,y,z)$ is a differentiable function, then $\nabla f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}$ is a vector field called the gradient vector field for f and we call f the potential function of ∇f.

Gradient vector fields are **VERY IMPORTANT** and we will use them a lot in future sections. A question we shall often ask ourselves is the following: Given a vector field $\vec{F}$, is it the gradient vector field of some function $f(x, y)$? Such a vector field is so important that we give them a special name.

Definition 2.2. Suppose $\vec{F} = \nabla f$ for some function f. Then we call $\vec{F}$ a **conservative vector field**.

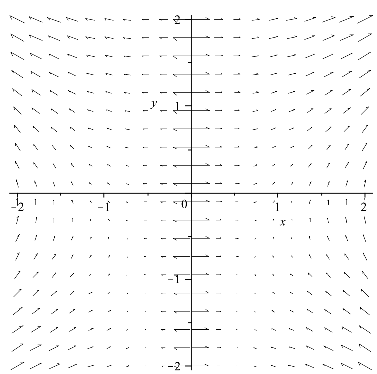
As noted above, a very important question we shall need to consider is the following: "Given a vector field $\vec{F}$, is it conservative?". Answering this is an important skill which needs to be developed i.e. we need to be able to determine if a vector field is conservative by finding a potential function (or showing that none exists). We finish with a couple of examples.

Example 2.3. (*i*) Sketch the gradient vector field of $f(x, y) = \ln(x) + yx^2$.

We have

$$\nabla f = \left(\frac{1}{x} + 2xy\right)\vec{i} + x^2\vec{j}.$$

The vector field looks like the following:



(*ii*) Is the vector field $\vec{F} = y\vec{i} + x\vec{j}$ conservative?

Integrate the first component with respect to x and we get $f(x, y) = xy + g(y)$ for some function g of y. Differentiating with respect to y, we get $f_y = x + g'(y) = x$ if $g'(y) = 0$. Therefore this is conservative with potential function $f(x, y) = xy$.

(*iii*) Is the vector field $\vec{F} = y\vec{i} - x\vec{j}$ conservative?

Integrate the first component with respect to x and we get $f(x, y) = xy + g(y)$ for some function g of y. Differentiating with respect to y, we get $f_y = x + g'(y) \neq x$ for any function if $g(y)$. Therefore this is not conservative.